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# A dynamical system approach to the construction of singular solutions of some degenerate elliptic equations

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## Abstract

We study the existence of singular separable solutions to the 2-dimensional quasilinear equation  $-\nabla \cdot (|\nabla u|^{p-2} \nabla u) + |u|^{q-1} u = 0$  under the form  $u(r, \theta) = r^{-\beta} \omega(\theta)$ . We obtain the full description of the set of such solutions by combining a 2-dimensional shooting method with a phase plane analysis approach.

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## 1. Introduction

Let  $1 < p < q + 1$  be real numbers, and  $(r, \theta) \in [0, \infty) \times S^1$  the polar coordinates in  $\mathbb{R}^2$ . The aim of this article is to give a complete description of the set of separable solutions of the degenerate elliptic equation with absorption

$$-\nabla \cdot (|\nabla u|^{p-2} \nabla u) + |u|^{q-1} u = 0 \quad (1.1)$$

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in  $\mathbb{R}^2 \setminus \{0\}$  in the form

$$u(r, \theta) = r^{-\beta} \omega(\theta), \quad (r, \theta) \in (0, \infty) \times S^1, \tag{1.2}$$

for some  $\beta \in \mathbb{R}$ . For homogeneity reasons

$$\beta = \beta_q = p/(q + 1 - p), \tag{1.3}$$

while  $\omega$  is a  $2\pi$ -periodic solution of

$$-\frac{d}{d\theta} [(\beta^2 \omega^2 + \omega'^2)^{\frac{p-2}{2}} \omega'] - a_{p,q} [\beta^2 \omega^2 + \omega'^2]^{\frac{p-2}{2}} \omega + \omega |\omega|^{q-1} = 0, \tag{1.4}$$

where  $a_{p,q} = \beta_q((\beta_q + 1)(p - 1) - 1)$ .

If  $p = 2$ , then  $\beta_q = 2/(q - 1)$  and (1.4) reduces to

$$-\omega'' - \left(\frac{2}{q-1}\right)^2 \omega + \omega |\omega|^{q-1} = 0, \tag{1.5}$$

which is the Euler–Lagrange equation of the functional  $E$  defined by

$$E(\omega) = \frac{1}{2} \int_{S^1} \left( \omega'^2 - \left(\frac{2}{q-1}\right)^2 \omega^2 + \frac{2}{q+1} |\omega|^{q+1} \right) d\theta. \tag{1.6}$$

Moreover, there exists an obvious first integral (usually called the Painlevé first integral), obtained here by a simple multiplication by  $\omega'$ ,

$$\frac{d}{d\theta} \left( \omega'^2 + \left(\frac{2}{q-1}\right)^2 \omega^2 - \frac{2}{q+1} |\omega|^{q+1} \right) = 0. \tag{1.7}$$

Those two observations make (1.5) easy to integrate. If we put  $\lambda = 4/(q - 1)^2$ , we replace (1.5) by the equation

$$-\omega'' - \lambda \omega + \omega |\omega|^{q-1} = 0, \tag{1.8}$$

and denote by  $\mathcal{E}_\lambda$  the set of its solutions. If  $k(\lambda)$  is the largest integer smaller than  $\sqrt{|\lambda|}$ , the following result is proved in [3].

**Theorem A.** *If  $\lambda \leq 0$ ,  $\mathcal{E}_\lambda = \{0\}$ . If  $0 < \lambda \leq 1$ ,  $\mathcal{E}_\lambda = \{0, \lambda^{1/(q-1)}, -\lambda^{1/(q-1)}\}$ . If  $\lambda > 1$ ,  $\mathcal{E}_\lambda$  has  $3 + k(\lambda)$  connected components:  $\mathcal{E}_\lambda^0, \mathcal{E}_\lambda^+, \mathcal{E}_\lambda^-,$  and  $\mathcal{E}_\lambda^k$  ( $1 \leq k \leq k(\lambda)$ ), where*

(i)  $\mathcal{E}_\lambda^0 = \{0\}$ ,  $\mathcal{E}_\lambda^+ = \{\lambda^{1/(q-1)}\}$ , and  $\mathcal{E}_\lambda^- = \{-\lambda^{1/(q-1)}\}$ ;

(ii) for each  $1 \leq k \leq k(\lambda)$ ,  $\mathcal{E}_\lambda^k$  is the set of all solutions to 1.8 with least anti-period  $\pi/k$ , and  $\mathcal{E}_\lambda^k = \{\omega(\cdot + \alpha) : \alpha \in S^1\}$ .

The above result can be interpreted via the bifurcation approach since when  $\lambda = \lambda_k = k^2$  the linearized problem

$$-\psi'' - \lambda\psi = 0, \tag{1.9}$$

is singular and any couple  $(\lambda_k, 0)$  is a bifurcation point from which a branch of solutions  $(\lambda, \omega)$  is issued. Moreover, any of these branches of solutions can be continued for  $\lambda > \lambda_k$ , and there exists no other solution.

When  $p \neq 2$ , (1.4) is not the Euler equation of any functional, and this makes the problem much more difficult to study. It is natural to introduce the set of singular separable  $p$ -harmonic functions, i.e. the set of solutions of

$$-\nabla \cdot (|\nabla v|^{p-2} \nabla v) = 0, \tag{1.10}$$

in  $\mathbb{R}^2 \setminus \{0\}$  which are written in the form

$$v(r, \theta) = r^{-\gamma} \phi(\theta), \quad (r, \theta) \in (0, \infty) \times S^1.$$

Then  $\phi$  is a  $2\pi$ -periodic solution of

$$-\frac{d}{d\theta} [(\gamma^2 \phi^2 + \phi'^2)^{\frac{p-2}{2}} \phi'] - b_{p,\gamma} [\gamma^2 \phi^2 + \phi'^2]^{\frac{p-2}{2}} \phi = 0, \tag{1.11}$$

where  $b_{p,\gamma} = \gamma(\gamma + 1)(p - 1) - 1$ . The set of solutions of (1.11) has been characterized by Kichenassamy and Véron [8] (and Kroll and Mazja [9] in the regular case). They proved

**Theorem B.** *Assume  $p > 1$ , then for each positive integer  $k$  there exists a  $\gamma_k \in \mathbb{R}$  and  $\phi_k : \mathbb{R} \rightarrow \mathbb{R}$  with least anti-period  $\pi/k$ , of class  $C^\infty$  such that*

$$v(x) = v(r, \theta) = r^{-\gamma_k} \phi_k(\theta), \tag{1.12}$$

is  $p$ -harmonic in  $\mathbb{R}^2 \setminus \{0\}$ ;  $\gamma_k$  is the positive root of

$$(\gamma + 1)^2 = (1 + 1/k)^2 (\gamma^2 + \gamma(p - 2) / (p - 1)). \tag{1.13}$$

The couple  $(\gamma_k, \phi_k)$  is unique, up to translation and homothety over  $\phi_k$ .

Let  $\mathcal{E}_{p,q}$  be the set of  $2\pi$ -periodic solutions of (1.4). We define

$$\ell_{p,q} = \left[ \left( \frac{p}{q + 1 - p} \right)^{p-1} \left( \frac{pq}{q + 1 - p} - 2 \right) \right]^{1/(q+1-p)}, \tag{1.14}$$

which exists only if  $(p - 2)q > 2(1 - p)$ , or equivalently  $(\beta_q + 1)(p - 1) > 1$ . The main result of this article is the following.

**Theorem 1.** *Assume  $q + 1 > p > 1$ . If  $\beta_q \leq (2 - p)/(p - 1)$ ,  $\mathcal{E}_{p,q} = \{0\}$ . If  $(2 - p)/(p - 1) < \beta_q \leq \gamma_1$ ,  $\mathcal{E}_{p,q} = \{0, \ell_{p,q}, -\ell_{p,q}\}$ . If  $\beta_q > \gamma_1$ , let  $k(q)$  be the largest integer such that  $\beta_q > \gamma_{k(q)}$ . Then  $\mathcal{E}_{p,q}$  has  $3 + k(q)$  connected components;  $\mathcal{E}_{p,q}^0$ ,  $\mathcal{E}_{p,q}^+$ ,  $\mathcal{E}_{p,q}^-$ , and  $\mathcal{E}_{p,q}^k$  ( $1 \leq k \leq k(q)$ ), where*

(i)  $\mathcal{E}_{p,q}^0 = \{0\}$ ,  $\mathcal{E}_{p,q}^+ = \{\ell_{p,q}\}$ , and  $\mathcal{E}_{p,q}^- = \{-\ell_{p,q}\}$ ;

(ii) *for each  $1 \leq k \leq k(q)$ ,  $\mathcal{E}_{p,q}^k$  is the set of solutions to (1.4), with least anti-period  $\pi/k$ , and  $\mathcal{E}_{p,q}^k = \{\phi(\cdot + \alpha) : \alpha \in S^1\}$ .*

The proof of this result is difficult and based upon two completely different points of view: a 2-dimensional shooting method, and a phase plane analysis. The shooting method consists in proving the existence of a positive solution of (1.1) in an angular sector  $\{(r, \theta) : r > 1, 0 < \theta < \theta_k\}$  of the 2-dimensional plane, and subject to Dirichlet conditions on the lateral boundary of the sector. Here  $\theta_k = \pi/k$  for some positive integer  $k$ . Thanks to the assumption on  $\beta$ , it will be proved that this solution is bounded from below and from above by two terms with the same decay order,  $C_i(\theta)r^{-\beta}$  for some functions  $C_i$  ( $i = 1, 2$ ). From this two-side estimate, these follows a precise asymptotic behavior (as  $r \rightarrow \infty$ ) which shows the existence of at least one positive solution of (1.4) on  $(0, \theta_k)$  vanishing at the end points. The non-existence is proved by the strong maximum principle. Surprisingly (and contrary to the semilinear case  $p = 2$ ), uniqueness cannot be obtained directly, neither from (1.1) nor from (1.4). Thus, we immerge this equation into a more general class of 2-dimensional differential systems and prove, by a phase plane analysis that the period of periodic solutions of such systems is a strictly monotone function of some shooting parameter. The dynamical systems approach for constructing solutions of non-linear equations is usually settled upon the invariant manifold theory: either its utilization is implicit as in [2] for constructing the very singular solution of the semilinear heat equation with absorption, or the theory is used in full as in [1,5,6] when studying ground states of a wide class of semilinear elliptic equations. In [7], this approach is combined with the use of the Mel'nikov function on invariant manifold in order to prove sharp asymptotics. The theory of internal isolated singularities is developed in [12–14].

**2. The shooting method**

We start this section with two key observations:

(A) By multiplying (1.4) by  $\omega$  and integrating over  $(0, 2\pi)$  we get

$$\int_0^\pi [\beta^2 \omega^2 + \omega'^2]^{\frac{p-2}{2}} \omega'^2 d\theta - a_{p,q} \int_0^\pi [\beta^2 \omega^2 + \omega'^2]^{\frac{p-2}{2}} \omega^2 d\theta + \int_0^\pi |\omega|^{q+1} d\theta = 0.$$

Thus, there is no non-trivial solution if  $a_{p,q} \leq 0$  or equivalently  $\beta_q \leq (2 - p)/(p - 1)$ . On the contrary, if  $a_{p,q} > 0$  there exists always two non-trivial constant solutions,  $\ell_{p,q}$ ,

defined by (1.14), and  $-\ell_{p,q}$ . Moreover, it is worth noticing that if  $p \geq 2$  it never occurs that  $\beta_q \leq (2 - p)/(p - 1)$ , therefore  $\mathcal{E}_{p,q}$  is never reduced to the zero function. In any case, we shall always assume  $a_{p,q} > 0$ .

(B) At the point  $\theta_0$  where  $\omega(\theta_0)$  vanishes,  $\omega'(\theta_0)$  is not zero. However, this is far from obvious except in the case  $p = 2$ , since the Cauchy–Lipschitz theorem does not hold at points where  $\omega$  and  $\omega'$  vanish. This fact will be the consequence of the following structure result.

**Proposition 1.** *Let  $q + 1 > p > 1$  such that  $a_{p,q} > 0$ . If  $\omega$  is a non-constant  $2\pi$ -periodic solution of (1.4), there exists a positive integer  $k$  such that  $\omega$  has least period  $2\pi/k$ . Moreover there exists  $\alpha \in S^1$  such that  $\omega_\alpha(\cdot) = \omega(\cdot - \alpha)$  vanishes at  $0$  and  $\pi/k$ , is positive on  $(0, \pi/k)$  and satisfies  $\omega'_\alpha(0) = -\omega'_\alpha(\pi/k) > 0$ .*

**Proof.** If  $\omega$  is a non-constant solution of (1.4) it is bounded from above (resp. from below) by  $\ell_{p,q}$  (resp.  $-\ell_{p,q}$ ). This follows from the maximum principle, since at a point of positive maximum (resp. negative minimum),  $\omega''$  exists (the equation is not degenerate) and the solution is locally  $C^2$  (see [10]). It is also a consequence of the fact that the function

$$U_M : x \mapsto \ell_{p,q} |x|^{-\beta_q}, \quad \forall x \neq 0, \tag{2.1}$$

is the maximal solution of (1.1) in  $\mathbb{R}^2 \setminus \{0\}$ , as it follows from the constructions in [4] (see also Vazquez’ universal a priori estimate [11]). Let us assume that  $\omega$  achieves positive values, and let  $\theta_1$  be such that  $\omega(\theta_1) = \max\{\omega(\theta) : \theta \in S^1\}$ . If  $\omega$  would keep a constant sign, it would have a positive minimum,  $\omega(\theta_2) < \ell_{p,q}$ . Since  $\omega''(\theta_2)$  exists at such a point (at this point, the equation is not degenerate since  $\omega(\theta_2) > 0$ ) it would be non-negative and (1.4) would imply

$$\omega^q(\theta_2) \geq a_{p,q} \beta^{p-2} \omega^{p-1}(\theta_2) \Rightarrow \omega(\theta_2) \geq \ell_{p,q},$$

a contradiction. Therefore,  $\omega$  is not always positive on  $S^1$ .

Let us denote by  $I_1(\theta_0, \tilde{\theta}_0)$ , the connected component containing  $\theta_1$  of the  $\theta \in S^1$ , where  $\omega(\theta) > 0$ . Thus,  $\omega$  is positive on  $(\theta_0, \tilde{\theta}_0)$  (identifying  $S^1$  with  $[0, 2\pi)$  we can assume  $\theta_0 < \tilde{\theta}_0$ ). Put

$$C_{\theta_0, \tilde{\theta}_0} = \{x = (r, \theta) \in \mathbb{R}^2 : r > 0, \theta \in (\theta_0, \tilde{\theta}_0)\}.$$

Then  $u(x) = u(r, \theta) = r^{-\beta} \omega(\theta)$  is a positive solution of (1.1) in the angular domain  $C_{\theta_0, \tilde{\theta}_0} \setminus \{0\}$ . In order to prove that  $\omega'(\theta_0) > 0$ , we consider  $a \in \mathbb{R}^2$  such that the open disk  $D_1(a)$ , of radius 1 and center  $a$  is included into  $C_{\theta_0, \tilde{\theta}_0}$  and tangent, at the point  $b$  and  $\tilde{b}$  to the two half-lines  $L_0 = \{(r, \theta) : r > 0, \theta = \theta_0\}$  and  $\tilde{L}_0 = \{(r, \theta) : r > 0, \theta = \tilde{\theta}_0\}$ . Although it plays no role in the sequel, the polar coordinates of  $a$  are  $[\sin((\theta_0 + \tilde{\theta}_0)/2)]^{-1}$  and  $(\theta_0 + \tilde{\theta}_0)/2$ . Let  $D_{(1+p)/2p}(a)$  be the disk of radius  $(1 + p)/2p$  and

center  $a$ . Since  $u$  is positive in  $D_1(a)$  its minimum on  $\partial D_{(1+p)/2p}(a)$  is positive. We denote it by  $\eta$ . Set

$$w_{A,B}(x) = A(1 - |x - a|) + B(1 - |x - a|)^2,$$

where  $A$  and  $B$  are positive parameters to be chosen such that

(i)  $w_{A,B} \leq \eta$  on  $\partial D_{(1+p)/2p}(a) \Leftrightarrow A(p - 1)/2p + B(p - 1)^2/4p^2 \leq \eta$ ,

(ii)  $-\nabla \cdot (|\nabla w_{A,B}|^{p-2} \nabla w_{A,B}) + w_{A,B}^q \leq 0$  in  $D_1(a) \setminus D_{(1+p)/2p}(a)$ . If we set  $\rho = x - a$ , this last inequality is equivalent to

$$-|A + 2B(1 - \rho)|^{p-2} [2(p - 1)B - \rho^{-1}(A + 2B(1 - \rho))] + (1 - \rho)^q (A + B(1 - \rho))^q \leq 0, \quad \forall \rho \in ((1 + p)/2p, 1).$$

Since

$$2(p - 1)B - \rho^{-1}(A + 2B(1 - \rho)) \geq \frac{2p(B - A)}{p + 1}, \quad \forall \rho \in ((1 + p)/2p, 1),$$

requirements (i) and (ii) are fulfilled as soon as we take  $0 < A < B$ , small enough. By the maximum principle,  $w_{A,B} \leq u$  in  $D_1(a) \setminus D_{(1+p)/2p}(a)$ . Both  $u$  and  $w_{A,B}$  vanish at the point  $b$ , and on  $L_0$  and  $\partial D_1(a)$ , respectively. Thus, if  $\nu$  denotes the outward normal derivative to  $D_1(a)$  at  $b$ ,

$$|b|^{-\beta} \omega'(\theta_0) = \frac{\partial u}{\partial \nu}(b) \leq \frac{\partial w_{A,B}}{\partial \nu}(b) = -A.$$

Here, we have used the fact that  $u$  is at least  $C^{1,\alpha}$  (for some  $0 < \alpha < 1$ ) by the regularity theory of degenerate elliptic equations, and so is  $\omega$ . Thus, the right and left derivatives of  $\omega$  at  $\theta_0$  coincide. Consequently,  $\beta^2 \omega(\theta_0^+)^2 + \omega'(\theta_0^+)^2 \neq 0$  and the Cauchy–Lipschitz existence and local uniqueness theorem applies at the point  $\theta_0$ . Since the equation is odd, the function  $\theta \mapsto -\omega(\theta_0 - \theta)$  is a solution of (1.4) in some interval  $(\theta_0 - \delta, \theta_0)$  on the left of  $\theta_0$ . Therefore,

$$\omega(\theta_0 - \theta) = -\omega(\theta), \quad \forall \theta \in (\theta_0 - \delta, \theta_0 + \delta) \subset (2\theta_0 - \tilde{\theta}_0, \tilde{\theta}_0). \tag{2.2}$$

Since  $\omega > 0$  on  $(\theta_0, \tilde{\theta}_0)$ , the equation is not degenerate on  $(\theta_0 - \delta, \theta_0]$ . This implies that the symmetry relation (2.2) holds on the interval  $[2\theta_0 - \tilde{\theta}_0, \tilde{\theta}_0]$ . Again, because the equation is not degenerate on this interval and  $\omega'(\theta_1) = 0$ , there holds

$$\omega(\theta) = \omega(2\theta_1 - \theta), \quad \forall \theta \in (0, \theta_1).$$

Consequently,  $\omega$  is  $2(\tilde{\theta}_0 - \theta_0)$ -periodic, with anti-period  $\tilde{\theta}_0 - \theta_0$ . The necessary and sufficient condition insuring  $\omega$  to be  $2\pi$ -periodic is that there exists a positive integer  $k$  such that  $\tilde{\theta}_0 - \theta_0 = \pi/k$ . Another consequence of the non-degeneracy of the

equation is that two solutions with the same period only differs by a phase coefficient  $\alpha$ . This implies the last statement.  $\square$

**Theorem 2.** *Let  $q + 1 > p > 1$ ,  $k \in \mathbb{N}_*$ ,  $\theta_k = \pi/k$ , then*

(i) *If  $\beta > \gamma_k$  there exists a positive solution  $\omega$  of (1.4) in  $(0, \theta_k)$  which vanishes at  $\theta = 0$  and  $\theta = \theta_k$ .*

(ii) *If  $\beta \leq \gamma_k$ , the only solution of (1.4) in  $(0, \theta_k)$  which vanishes at  $\theta = 0$  and  $\theta = \theta_k$  is the zero function.*

**Proof.** For  $R \geq 0$ , we set

$$C_{0,\theta_k}^R = \{x = (r, \theta): r > R, 0 < \theta < \theta_k\},$$

and  $C_{0,\theta_k} = C_{0,\theta_k}^0$ . If a solution  $\omega$  of (1.4) in  $(0, \theta_k)$  vanishing at the two end points exists, the function

$$(r, \theta) \mapsto u_k(r, \theta) = r^{-\beta} \omega(\theta),$$

is a solution of (1.1) in  $C_{0,\theta_k}^R$  for any  $r > R$  vanishing on the lateral boundary

$$\partial_L C_{0,\theta_k}^R = \{x = (r, 0): r > R\} \cup \{x = (r, \theta_k): r > R\}.$$

*Step 1. Construction of an approximate solution:* We define the function  $h \in C(\mathbb{R}^2)$  by

$$h(x) = \begin{cases} 2 - |x| & \text{if } |x| \leq 2, \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

For  $n > R \geq 1$  we set  $C_{0,\theta_k}^{R,n} = C_{0,\theta_k}^R \setminus \overline{C_{0,\theta_k}^n}$ . Let  $\varepsilon > 0$  to be chosen, and  $v_n$  the solution (obtained by minimization) of

$$-\nabla \cdot (|\nabla v_n|^{p-2} \nabla v_n) + |v_n|^{q-1} v_n = 0 \quad \text{in } C_{0,\theta_k}^{1,n},$$

$$v_n = \varepsilon h \quad \text{on } \partial C_{0,\theta_k}^{1,n}. \tag{2.3}$$

The function  $v_n$  is non-negative. We choose  $\varepsilon$  such that

$$\varepsilon h(x) \leq U_M(x), \quad \forall x \in \mathbb{R}^2.$$

(remember that  $U_M$ , defined by (2.1) is the maximal solution of (1.1) in  $\mathbb{R}^2 \setminus \{0\}$ ). By monotonicity and the maximum principle, we have

$$1 < n_1 < n_2 \Rightarrow v_{n_1} \leq v_{n_2} \leq U_M \quad \text{in } C_{0,\theta_k}^{1,n_1}.$$

When  $n$  tends to infinity,  $u_n$  increases and converges to some  $u$  which is a positive solution of (1.1) in  $C_{0,\theta_k}^1$ , with boundary value  $\varepsilon h$  on  $\partial C_{0,\theta_k}^1$ . Moreover,

$$u(x) \leq U_M(x) = \ell_{p,q}|x|^{-\beta} \quad \text{in } C_{0,\theta_k}^1.$$

Notice that the decay at infinity and the monotone operators theory are enough to ensure the uniqueness of  $u$ .

*Step 2.* Estimate from below: Let  $\phi_k$  be a solution of (1.11) with corresponding exponent  $\gamma = \gamma_k$ , normalized by

$$0 \leq \phi_k \leq \max_{0,\theta_k} \phi_k = 1.$$

We set  $\sigma = \beta/\gamma_k$ . Then  $\sigma > 1$ . We claim that, for  $\eta > 0$  small enough,

$$(r, \theta) \mapsto V_\eta(r, \theta) = \eta r^{-\beta} \phi_k^\sigma(\theta),$$

is a non-negative subsolution of (1.1) in  $C_{0,\theta_k}^1$  which vanishes on the lateral boundary  $\partial_L C_{0,\theta_k}^1$ . If we denote

$$\mathcal{P}(V) = -\nabla \cdot (|\nabla V|^{p-2} \nabla V) + |V|^{q-1} V,$$

and

$$\mathcal{D}(\psi) = -\frac{d}{d\theta} [(\beta^2 \psi^2 + \psi'^2)^{\frac{p-2}{2}} \psi'] - a_{p,q} [\beta^2 \psi^2 + \psi'^2]^{\frac{p-2}{2}} \psi + |\psi|^{q-1} \psi,$$

then

$$\mathcal{P}(V_\eta) = r^{-q\beta} \mathcal{D}(\eta \phi_k^\sigma).$$

Put  $\psi = \eta \phi_k^\sigma$ . By a straightforward computation one obtains

$$(\beta^2 \psi^2 + \psi'^2)^{\frac{p-2}{2}} = \eta^{p-2} \sigma^{p-2} \phi_k^{(\sigma-1)(p-2)} (\gamma_k^2 \phi_k^2 + \phi_k'^2)^{\frac{p-2}{2}},$$

and

$$\begin{aligned} \frac{d}{d\theta} [(\beta^2 \psi^2 + \psi'^2)^{\frac{p-2}{2}} \psi'] &= \eta^{p-1} \sigma^{p-1} \frac{d}{d\theta} [\phi_k^{(\sigma-1)(p-1)} (\gamma_k^2 \phi_k^2 + \phi_k'^2)^{\frac{p-2}{2}} \phi_k'] \\ &= \eta^{p-1} \sigma^{p-1} \phi_k^{(\sigma-1)(p-1)} \frac{d}{d\theta} [(\gamma_k^2 \phi_k^2 + \phi_k'^2)^{\frac{p-2}{2}} \phi_k'] \\ &\quad + (\sigma - 1)(p - 1) \eta^{p-1} \sigma^{p-1} \phi_k^{(\sigma-1)(p-1)-1} (\gamma_k^2 \phi_k^2 + \phi_k'^2)^{\frac{p-2}{2}} \phi_k'^2. \end{aligned}$$

Since

$$-\frac{d}{d\theta} [(\gamma_k^2 \phi_k^2 + \phi_k'^2)^{\frac{p-2}{2}} \phi_k'] = b_{p,\gamma_k} (\gamma_k^2 \phi_k^2 + \phi_k'^2)^{\frac{p-2}{2}} \phi_k,$$

with  $b_{p,\gamma_k} = \gamma_k((\gamma_k + 1)(p - 1) - 1)$ , it follows that

$$\eta^{1-p} \mathcal{D}(\psi) = \eta^{q+1-p} \phi_k^{q\sigma} + \sigma^{p-2} \phi_k^{(\sigma-1)(p-1)-1} (\gamma_k^2 \phi_k^2 + \phi_k'^2)^{\frac{p-2}{2}} [(\sigma b_{p,\gamma_k} - b_{p,q}) \phi_k^2 - \sigma(\sigma - 1)(p - 1) \phi_k'^2].$$

But  $\sigma b_{p,\gamma_k} - b_{p,q} = \beta(\gamma_k - \beta)(p - 1) = -\gamma_k^2 \sigma(\sigma - 1)(p - 1)$ . Therefore,

$$\begin{aligned} \eta^{1-p} \mathcal{D}(\psi) &= \eta^{q+1-p} \phi_k^{q\sigma} - (p - 1)(\sigma - 1) \sigma^{p-1} \phi_k^{(\sigma-1)(p-1)-1} (\gamma_k^2 \phi_k^2 + \phi_k'^2)^{\frac{p}{2}} \\ &\leq \eta^{q+1-p} \phi_k^{q\sigma} - (p - 1)(\sigma - 1) \sigma^{p-1} \phi_k^{\sigma(p-1)}. \end{aligned} \tag{2.4}$$

Since  $\sigma > 1$ , the right-hand side of (2.4) is non-positive for  $\eta$  small enough. We can also impose  $\eta \phi_k \leq \varepsilon$  in order to have  $V_\eta \leq u(x)$  if  $|x| = 1$ . By the maximum principle,  $V_\eta$  is dominated by  $u$  in  $C_{0,\theta_k}^1$ . This implies

$$\eta \phi_k^\sigma(x/|x|) \leq |x|^\beta u(x) \leq \ell_{p,q} \quad \text{in } C_{0,\theta_k}^1. \tag{2.5}$$

*Step 3. Asymptotic behavior:* For  $R > 0$ , we define  $u_R$  by  $u_R = R^\beta u(Rx)$ . The function  $u_R$  satisfies (1.1) in  $C_{0,\theta_k}^{1/R}$ . By the degenerate elliptic equation regularity theory, the set of functions  $\{u_R\}$  remains bounded in the  $C_{\text{loc}}^{1,\alpha}$ -topology of  $\overline{C_{0,\theta_k}} \setminus \{0\}$ . Since

$$\frac{d}{dR} (R^\beta (2 - R|x|)_+^\beta) = \beta R (2 - R|x|)_+^{\beta-1} (2 - 2R|x|) \leq 0 \quad \text{for } |x| \geq 1/R,$$

there holds

$$R'^\beta (2 - R'|x|)_+^\beta \leq R^\beta (2 - R|x|)_+^\beta \quad \text{for } |x| \geq 1/R,$$

for  $0 < R < R'$ . Because  $h(x) = (2 - |x|)_+$ , it follows by the maximum principle

$$R' \geq R \Rightarrow u_{R'} \leq u_R \in C_{0,\theta_k}. \tag{2.6}$$

Thus, there exists a function  $u^*$  such that  $u_R$  decreases and converges to  $u^*$  as  $R \rightarrow \infty$  in  $C_{\text{loc}}^1(\overline{C_{0,\theta_k}} \setminus \{0\})$ . The function  $u^*$  is a solution of (1.1) in  $C_{0,\theta_k}$  which vanishes on  $\partial_L C_{0,\theta_k}$ . Because of (2.5),  $u^*$  satisfies

$$\eta \phi_k^\sigma(x/|x|) \leq |x|^\beta u^*(x) \leq \ell_{p,q} \quad \text{in } C_{0,\theta_k}. \tag{2.7}$$

Finally,

$$\lim_{R \rightarrow \infty} R^\beta u(Rr, \theta) = u^*(r, \theta) = r^{-\beta} \lim_{R \rightarrow \infty} (Rr)^\beta u(Rr, \theta) = r^{-\beta} u^*(1, \theta).$$

If we define  $\omega$  by  $\omega(\theta) = u^*(1, \theta)$ , then  $u^*(r, \theta) = r^{-\beta}\omega(\theta)$ , and

$$\eta\phi_k^\sigma(\theta) \leq \omega(\theta) \leq \ell_{p,q}, \quad \forall \theta \in [0, \theta_k]. \tag{2.8}$$

This implies that  $\omega$  is a positive solution of (1.4) on  $(0, \theta_k)$  which vanishes at the two end points.

*Step 4. Non-existence:* Let us assume  $\beta \leq \gamma_k$  and there exists a positive solution  $\omega$  of (1.4) in  $(0, \theta_k)$  vanishing at the two end points. In this case,  $\sigma = \beta/\gamma_k \leq 1$ . We still define  $V_\eta$  by

$$V_\eta(r, \theta) = \eta r^{-\beta} \phi_k^\sigma(\theta),$$

where  $\eta > 0$  and obtain, with  $\psi = \eta\phi_k^\sigma$ ,

$$\eta^{1-p} \mathcal{D}(\psi) = \eta^{q+1-p} \phi_k^{q\sigma} - (p-1)(\sigma-1)\sigma^{p-1} \phi_k^{(\sigma-1)(p-1)-1} (\gamma_k^2 \phi_k^2 + \phi_k'^2)^{\frac{p}{2}} > 0.$$

We choose  $\eta = \eta_0 > 0$  as the smallest parameter such that  $\eta\phi_k^\sigma \geq \omega$ . Notice that this is possible since both  $\omega$  and  $\phi_k^\sigma$  are  $C^1$  and positive in the interval  $(0, \theta_k)$  on the end points of which  $\phi_k'$  does not vanish.

*Case 1:* There exists  $\theta_0 \in (0, \theta_k)$  such that

$$\eta\phi_k^\sigma(\theta) \geq \omega(\theta), \quad \forall \theta \in [0, \theta_k] \quad \text{and} \quad \eta\phi_k^\sigma(\theta_0) = \omega(\theta_0).$$

Notice that the above configuration always holds if  $\sigma < 1$ . By the mean value theorem,

$$(\beta^2 \psi^2 + \psi'^2)^{(p-2)/2} \psi' - (\beta^2 \omega^2 + \omega'^2)^{(p-2)/2} \omega' = a(\psi' - \omega') + b(\psi - \omega),$$

where

$$b = (p-2)(\beta^2(\omega + t(\psi - \omega))^2 + (\omega' + t(\psi' - \omega'))^2)^{(p-4)/2} \times (\omega + t(\psi - \omega))(\omega' + t(\psi' - \omega'))$$

and

$$a = (p-2)(\beta^2(\omega + t(\psi - \omega))^2 + (\omega' + t(\psi' - \omega'))^2)^{(p-4)/2} \times (\omega + t(\psi - \omega))(\omega' + t(\psi' - \omega'))^2 + (\beta^2(\omega + t(\psi - \omega))^2 + (\omega' + t(\psi' - \omega'))^2)^{(p-2)/2},$$

for some  $t \in (0, 1)$ . Moreover,

$$\psi(\theta_0) = \omega(\theta_0) = \Theta > 0 \quad \text{and} \quad \psi'(\theta_0) = \omega'(\theta_0) = A.$$

Therefore,

$$b(\theta_0) = (p - 2)(\beta^2 \Theta^2 + A^2)^{(p-4)/2} \Theta A$$

and

$$a(\theta_0) = (\beta^2 \Theta^2 + A^2)^{(p-4)/2} (\beta^2 \Theta^2 + (p - 1)A^2).$$

Thus,  $a(\theta_0) > 0$ , and this property holds in a neighborhood of  $\theta_0$ . Since the two equations are not degenerate, the functions  $a$  and  $b$  are  $C^1$  in a neighborhood of  $\theta$ . Similarly,

$$(\beta^2 \psi^2 + \psi'^2)^{(p-2)/2} \psi - (\beta^2 \omega^2 + \omega'^2)^{(p-2)/2} \omega = c(\psi' - \omega') + d(\psi - \omega),$$

and

$$\psi^q - \omega^q = e((\psi - \omega)),$$

for some bounded  $C^1$  functions  $c$ ,  $d$  and  $e$ . From this we have

$$\mathcal{D}(\psi) - \mathcal{D}(\omega) = -\frac{d}{d\theta} (a(\psi' - \omega') + b(\psi - \omega)) - a_{p,q} c(\psi' - \omega') + (d + e)(\psi - \omega).$$

Since  $\mathcal{D}(\psi) - \mathcal{D}(\omega) \geq 0$ ,

$$-\frac{d}{d\theta} (a(\psi' - \omega')) - (a_{p,q} c + b)(\psi' - \omega') + (d + e - b')(\psi - \omega)_+ \geq 0, \tag{2.9}$$

holds, with  $\psi - \omega \geq 0$  and  $(\psi - \omega)(\theta_0) = 0$ . By the strong maximum principle  $\psi - \omega \equiv 0$  in a neighborhood of  $\theta_0$ . Therefore, the set of points where  $\psi$  and  $\omega$  coincide is open in  $(0, \theta_0)$ . Since it is closed by continuity, it implies  $\psi - \omega \equiv 0$  in the whole interval  $(0, \theta_0)$ . Contradiction.

*Case 2:*  $\sigma = 1$  and there exists  $\eta > 0$  such that

$$\begin{aligned} \eta \phi_k(\theta) > \omega(\theta), \quad \forall \theta \in (0, \theta_k) \quad \text{and} \quad \eta \phi_k'(0) = \omega'(0) \quad \text{or} \\ \eta \phi_k'(\theta_k) = \omega'(\theta_k). \end{aligned}$$

We proceed as above, using the fact that  $\phi_k'$  does not vanish at the end points. Therefore,  $\psi - \omega \geq 0$  in a neighborhood of 0 (or  $\theta_k$  similarly). The inequality (2.9) holds with a strongly elliptic operator. Since  $(\psi - \omega)(\theta_0) = 0$ , the Hopf boundary lemma implies  $(\psi' - \omega')(\theta_0) > 0$ , which contradicts the tangency of the two graphs at  $\theta = 0$ .  $\square$

The proof of uniqueness of  $\omega$  will be obtained by the phase plane analysis developed in the next section.

### 3. The phase plane analysis

#### 3.1. Dynamical system and critical points

In this section, we will consider the following more general ordinary differential equation (ODE) in the real variable  $t$ :

$$-\frac{d}{dt}[(\beta^2\omega^2 + \omega'^2)^{\frac{p-2}{2}}\omega'] - \alpha[\beta^2\omega^2 + \omega'^2]^{\frac{p-2}{2}}\omega + g(\omega) = 0, \tag{3.1}$$

where  $\alpha > 0$  and  $g$  is a differentiable and odd function that satisfies the following hypotheses:

- (H1)  $\alpha\beta^{p-2} \in \text{Im}(F)$ , where  $F(x) := \frac{g(x)}{x|x|^{p-2}}$ ,
- (H2)  $F$  is a strictly increasing function.

In order to transform the ODE into a dynamical system put  $x = \omega$  and  $y = \omega'$ , then we get

$$(\mathcal{S}) \begin{cases} x' = P(x, y) := y, \\ y' = Q(x, y) := \frac{g(x)[\beta^2x^2 + y^2]^{\frac{4-p}{2}} - \alpha\beta^2x^3 - (\alpha + (p-2)\beta^2)xy^2}{(p-1)y^2 + \beta^2x^2}. \end{cases}$$

The only singular point of the system is the origin  $(0, 0)$ . Therefore, at any point in  $\mathbb{R}^2 \setminus \{0, 0\}$  the Cauchy–Lipschitz local existence and uniqueness theorem applies. A direct calculation shows that the critical points of  $(\mathcal{S})$ , are given by

$$y = 0 \quad \text{and} \quad F(x) = \alpha\beta^{p-2} > 0. \tag{3.2}$$

Since  $F$  is strictly increasing and even, we have then two critical points,  $(-c, 0)$  and  $(c, 0)$ , where  $c = F^{-1}(\alpha\beta^{p-2})$ . The linearized system at a point  $(x, 0)$ , is given by the matrix

$$A(x) := \begin{bmatrix} 0 & 1 \\ a(x) & 0 \end{bmatrix},$$

where

$$a(x) := \beta^{2-p}[-\alpha\beta^{p-2} + g'(x)|x|^{2-p} - (p-2)g(x)|x|^{-p}x].$$

Note that the condition (H2) on  $F$ , for  $x > 0$ , is equivalent to  $xg'(x) > (p-1)g(x)$  and  $a(x) > \beta^{2-p}[x^{1-p}g(x) - \alpha\beta^{p-2}]$ . Thus,  $a(c) > 0$ .

**Remark 1.** Define  $H(x) := Q(x, 0) = x\beta^{2-p}[F(x) - F(c)]$ , we have

$$H'(x) = x\beta^{2-p}[F(x) - F(c) + xF'(x)],$$

and then by the condition (H2) on  $F$ , we see that  $H(x)$  is strictly negative for  $0 < x < c$ , and strictly positive and increasing for  $x > c$ .

The eigenvalues  $\lambda_1$  and  $\lambda_2$  of  $A(c)$  are given by the following algebraic system:

$$\begin{cases} \lambda_1 + \lambda_2 = 0, \\ \lambda_1\lambda_2 = -a(c), \end{cases}$$

and the associated eigenvectors are

$$\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

Note that in this case, both  $(c, 0)$  and  $(-c, 0)$ , are saddle points.

### 3.2. Qualitative study of the dynamical system

**Notation 1.** We will use the following notation:  $\mathcal{O}(x_0, y_0)$  is the orbit  $\{(x(t), y(t))\}$ , that passes through the point  $(x_0, y_0)$  at  $t = 0$ .

**Lemma 1.** Concerning the dynamical system  $(\mathcal{S})$ , we have

- (a) *The  $x$ - and  $y$ -axis are axes of symmetry, and so the origin is a center of symmetry.*
- (b) *Orbit of solutions of any autonomous system is invariant by time shift.*

**Proof.** The second point is obvious and to see the first point (a), consider the following applications:

$$\Phi_x : (t, x, y) \mapsto (-t, -x, y), \quad \Phi_y : (t, x, y) \mapsto (-t, x, -y).$$

It is clear that,  $(\mathcal{S})$  is invariant under  $\Phi_x$  and  $\Phi_y$ . Since the Cauchy–Lipschitz theorem applies, any orbit which intersects the  $x$ -axis (always perpendicularly) is symmetric with respect to the  $y$ -axis, and similarly by exchanging the role of the two axes.  $\square$

This will give directly

**Lemma 2.** *Any orbit which intersects both  $x$ - and  $y$ -axis is a closed orbit.*

**Proof.** Without loss of generality (see 1(b)), we can assume that the orbit starts at  $(0, y_0)$ , at time  $t = 0$ , Applying in this order the applications  $\Phi_y, \Phi_x$ , and  $\Phi_y$ , we get the result.  $\square$

Now, we define the following subsets:

$$I_1 = \{y_0 > 0: \mathcal{O}(0, y_0) \cap \mathcal{O}x \neq \emptyset \text{ (finite time)}\},$$

$$I_2 = \{y_0 > 0: \mathcal{O}(0, y_0) \text{ satisfies } (\mathcal{P})\},$$

where the property  $(\mathcal{P})$  means

there exists  $\epsilon_0 > 0$  such that  $y(t) \geq \epsilon_0, \forall t \in \mathbb{R}$ .

**Lemma 3.** *The set  $I_1$  is a non-empty interval.*

**Proof.** *Step 1.* The set  $I_1$  is not empty. Let  $x_0 \in (0, c)$ , and consider the orbit  $\mathcal{O}(x_0, 0)$ . We will show that the trajectory intersects the  $y$ -axis at finite time. Using  $(\mathcal{S})$  we have

$$x'(0) = 0 \quad \text{and} \quad y'(0) = g(x_0)(\beta x_0)^{2-p} - \alpha x_0 < 0,$$

and then,  $x$  and  $y$  are decreasing from  $x_0$  and  $0$ , respectively. Note that the orbit cannot cross again the  $x$ -axis or the origin, since  $y' < 0$ .

Thus, if the orbit does not intersect the  $y$ -axis the unique possibility is to have a vertical asymptote. In this case, we must have

$$x'(t) \rightarrow 0 \quad \text{and} \quad y(t) \rightarrow -\infty \text{ as } t \rightarrow +\infty.$$

From the equation,  $x'(t) = y(t)$ , we get  $y(t) \rightarrow 0$  as  $t \rightarrow +\infty$ , which is not possible.

Therefore, there exists  $t_0 > 0$ , such that  $x(t_0) = 0$ , and so  $-y(t_0) \in I_1$ .

*Step 2.* The set  $I_1$  is an interval. Let  $y_0 \in I_1$ , and let us prove that  $]0, y_0] \subseteq I_1$ . Indeed, if  $0 < y_1 < y_0$ , then the orbit  $\mathcal{O}(0, y_1)$  cannot intersect the orbit  $\mathcal{O}(0, y_0)$ , and it cannot tend to the origin or any equilibrium, because  $x' > 0$ . So necessarily it intersects the  $x$ -axis.  $\square$

**Remark 2.** By Lemma 2, if  $y_0 \in I_1$ , then the point  $(0, y_0)$  belongs to a periodic orbit.

**Lemma 4.** *Every periodic orbit intersects the  $x$ -axis in the interval  $(-c, c)$ .*

**Proof.** By the Poincaré–Bendixon theorem, the bounded open domain of  $\mathbb{R}^2$  bordered by a closed orbit contains at least a stationary point,  $(-c, 0)$  or  $(c, 0)$ , or the singular point  $(0, 0)$ . Thus, any periodic trajectory intersects the  $x$ -axis. Assume that  $x_1 > c$  is the intersection of a closed orbit with the  $x$ -axis. For  $t > 0$  small enough, the solution  $t \mapsto (x(t), y(t))$  with initial data  $(x_1, 0)$  satisfies  $x'(t) > 0$  and  $y'(t) > 0$  (by Remark 1). Therefore, the  $x$ -coordinates are increasing as long as the trajectory

belongs to the upper half-plane  $y > 0$ . This implies that the abscissa  $x_2$  of the second intersection point  $(x_2, 0)$  with the  $x$ -axis satisfies  $x_2 > x_1$ . Since the  $x$ -axis is an axis of symmetry, the whole trajectory  $\mathcal{O}(x_1, 0)$  is obtained by reflection through this axis, and the bounded open domain of  $\mathbb{R}^2$  that it encloses contains no stationary or singular point, a contradiction.  $\square$

**Lemma 5.** *The set  $I_2$  is a non-empty open interval.*

**Proof.** Consider  $x_0 = c$  and  $y_0 > 0$ , and the associated orbit  $\mathcal{O}(x_0, y_0)$ , with for some  $\tau$ ,  $x(\tau) = c$  and  $y(\tau) = y_0$ . Since  $x'(\tau) = y_0$ ,  $t \mapsto x(t)$  is increasing for  $t \geq \tau$ .

Since for all  $x > c$ ,  $Q(x, 0) > 0$  and  $\lim_{x \rightarrow \infty} Q(x, 0) = +\infty$ , the orbit could not intersect the  $x$ -axis and the limit of  $y$  could not be zero as  $t \rightarrow \infty$ . We cannot have a horizontal asymptote: otherwise,  $\lim_{t \rightarrow \infty} y(t) = r > 0$  so this means that  $\lim_{x \rightarrow \infty} y'(t) = 0$ , but in this case  $\lim_{x \rightarrow \infty} x(t) = \infty$  and  $0 = \lim_{x \rightarrow \infty} Q(x, r) = \infty$ . Thus, necessarily  $\lim_{t \rightarrow +\infty} y(t) = +\infty$ . Therefore, this orbit is bounded below, for  $t \geq \tau$ , by some  $\epsilon_0$ .

Next we will show that this orbit  $\mathcal{O}(x_0, y_0)$  intersects the  $y$ -axis. For this, consider the orbit  $\tilde{\mathcal{O}}_r(x_0, -y_0)$ , with components  $(\tilde{x}(t), \tilde{y}(t))$  with  $\tilde{x}(0) = x_0$  and  $\tilde{y}(0) = -y_0$ . By the argument above  $\tilde{y}(t) \leq -\epsilon_0$  for  $t \leq 0$ , so  $\tilde{x}(t)$  decreases from  $c$ . Notice again that this orbit  $\mathcal{O}(x_0, y_0)$  could not cross the  $x$ -axis or the origin. By the same argument of the proof of Lemma 3 this orbit does not have a vertical asymptote, so it must intersect the  $y$ -axis in some  $y_1 < 0$  and so  $-y_1 \in I_2$ .

Now to see that  $I_2$  is an interval, let  $y_0 \in I_2$  and  $y_2 > y_0$ . Then immediately  $y_2 \in I_2$ , since the orbit  $\mathcal{O}(0, y_2)$  cannot intersect  $\mathcal{O}(0, y_0)$ .

Now, we are going to prove that  $I_2$  is an open interval. Indeed, let  $y_0 \in I_2$ , and consider the orbit  $\mathcal{O}(0, y_0)$  with components  $(x(t), y(t))$ . We know that  $x$  is always increasing from 0. There exists then  $t_1$ , such that,  $x(t_1) = c$  and  $y(t_1) > 0$ . Let  $\mathcal{O}_r(\bar{x}, \bar{y})$  be the orbit where,  $\bar{x} = c$  and  $y(t_1) > \bar{y} > 0$ . By the same argument,  $\mathcal{O}_r(\bar{x}, \bar{y})$  crosses the  $y$ -axis at  $(0, y_1)$ , for some  $y_0 > y_1$ , and therefore  $y_1 \in I_2$  and so  $y_0 \in (y_1, +\infty) \subset I_2$ .  $\square$

**Remark 3.** It is easy to see that  $I_1$  is an interval bounded above by any element of  $I_2$ . Therefore, we denote

$$b := \sup I_1 < +\infty.$$

**Lemma 6.** *For every  $x_0 \in (-c, c) \setminus \{0\}$ , there exists a unique periodic orbit such that  $(x_0, 0)$  belongs to this orbit.*

**Proof.** By symmetry and same analysis as in the proof of Lemma 3.  $\square$

**Lemma 7.**  *$I_1$  is an open interval.*

**Proof.** We know that  $I_1$  is an interval. We need to show that  $b \notin I_1$ . We proceed by contradiction and assume that  $b \in I_1$ . Then there exists, a first time,  $t_0 > 0$ , such that

$y(t_0) = 0$ . By Lemma 4, we know that  $x(t_0) \in (0, c)$ . So for any  $x(t_0) < x_1 < c$ , and by the last lemma,  $(x_1, 0)$  belongs to a periodic orbit  $\mathcal{O}(x_1, 0)$ . This periodic orbit intersects the  $y$ -axis at some  $y(t_1) > b$ , which is impossible since  $b$  is the supremum of  $I_1$ .  $\square$

**Proposition 2.** *There exists one and only one heteroclinic orbit which connects the points  $(-c, 0)$  and  $(c, 0)$  in the upper half-plane, and one heteroclinic orbit that connects  $(c, 0)$  and  $(-c, 0)$  in the lower half-plane.*

**Proof.** Consider the orbit  $\mathcal{O}(0, b)$ ,  $b$  as above, and let us show that

$$\lim_{t \rightarrow +\infty} x(t) = c \quad \text{and} \quad \lim_{t \rightarrow +\infty} y(t) = 0.$$

Using the fact that  $b \notin I_1$ , we obtain,  $\lim_{t \rightarrow +\infty} x(t) \geq c$ . Suppose  $\lim_{t \rightarrow +\infty} x(t) > c$ . Then there exists  $t_0$  such that,  $x(t_0) = c$  and  $y(t_0) > 0$ . Proceeding as in the proof of Lemma 5, we get  $b \in I_2$ , which is impossible since  $I_2$  is open and  $I_1 \cup \overline{I_2} = (0, +\infty)$ . Thus,  $\lim_{t \rightarrow \infty} x(t) = c$ . From  $x' = y$ , we obtain

$$\lim_{t \rightarrow +\infty} y(t) = 0.$$

The orbit in the lower half-plane is obtained by symmetry.  $\square$

**Remark 4.** The heteroclinic orbit in the upper half-plane is the unstable (resp. stable) trajectory of the point  $(-c, 0)$  (resp.  $(c, 0)$ ).

### 3.3. Variation of the period

In this section, we will consider a general dynamical system

$$(\mathcal{S}') \begin{cases} x' = F(x, y), \\ y' = G(x, y), \end{cases}$$

where  $F$  and  $G$  are  $C^1$  functions in  $\mathbb{R}^2 \setminus \{(0, 0)\}$ , in which region the system has no equilibrium. We assume that  $F$  (resp.  $G$ ) is odd with respect to  $y$  (resp.  $x$ ) and even with respect to  $x$  (resp.  $y$ ). As in Lemma 2, these equivariance properties imply that any orbit of  $\mathcal{S}'$  which intersects both  $x$ - and  $y$ -axis is closed. We assume also that there exists an open interval  $(0, b)$  ( $b > 0$ ) of the positive  $y$ -axis such that for any  $y \in (0, b)$ , the positive trajectory of  $(\mathcal{S}')$  through  $(0, y)$  enters the region  $Q_+ = \{(x, y) \in \mathbb{R}^2: x > 0, y > 0\}$  and escapes from  $Q_+$  in crossing the positive  $x$ -axis in finite time. Let us denote by  $T(y)$  the first time such a trajectory intersects the  $x$ -axis. The orbit through  $(0, y)$  is periodic and symmetric with respect to the coordinates axis, therefore, it is  $4T(y)$ -periodic. If  $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ , we denote by  $M_{(x,y)}(t) = (x(t), y(t))$  the maximal solution of  $(\mathcal{S}')$  with initial data  $(x, y)$ . Since the function  $T$

is continuous, the set

$$\mathcal{R} = \bigcup_{0 < y < b} \bigcup_{0 < t < T(y)} \{M_{(0,y)}(t)\},$$

is a non-empty open subset of  $Q_+$ .

The aim of this section, under some hypothesis on  $F$  and  $G$ , is to prove a general result on the monotonicity of the period as function of the  $y$  coordinates of the initial data.

**Notation 2.** For all  $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$  denote by  $V$  the vector  $(F(x, y), G(x, y))$  and for  $\lambda \neq 0$ , denote by  $V_\lambda$  the vector  $(F(\lambda x, \lambda y), G(\lambda x, \lambda y))$ .

The main assumption on  $V$  is the following:

(H3) The functions  $F$  and  $G$  are, respectively, *positively homogeneous of degree 1* and *superpositively homogeneous of degree 1* in  $\mathcal{R}$ , which is

$$F(\lambda x, \lambda y) = \lambda F(x, y) \quad \text{and} \quad G(\lambda x, \lambda y) > \lambda G(x, y) \tag{3.3}$$

for all  $(x, y) \in \mathcal{R}$  and all  $\lambda > 1$  such that  $(\lambda x, \lambda y) \in \mathcal{R}$ .

(H4) The function  $F$  remains positive in  $\mathcal{R}$ .

**Remark 5.** Concerning the assumption on  $G$ , we could have equality:  $G(\lambda x, \lambda y) = \lambda G(x, y)$  on a discrete subset of each orbit.

In order to show that the function  $T$  defined above is strictly increasing, let us consider  $\mathcal{O}(0, y_0)$  an orbit with components  $M(t) = M_{(0,y_0)}(t) = (x(t), y(t))$ , such that  $(x(0), y(0)) = (0, y_0) \in \mathcal{R}$ ,  $\lambda > 1$ , such that  $\lambda\mathcal{O}$ , the homothetic of  $\mathcal{O}$  with components  $(\lambda x(t), \lambda y(t))$  still in  $\mathcal{R}$ . Notice that  $\lambda\mathcal{O}$  and  $\mathcal{O}$  have the same period.

Finally, consider the orbit  $\mathcal{O}_\lambda$  with components  $M_\lambda(t) = (x_\lambda(t), y_\lambda(t)) \in \mathcal{R}$ ,  $\lambda > 1$ , such that,  $x_\lambda(0) = 0$  and  $y_\lambda(0) = \lambda y_0 \in (0, b)$ .

**Remark 6.** The hypothesis on  $F$  and  $G$  have the following geometrical interpretations:

(i) For all  $(x, y) \in \mathcal{R}$ ,  $(\lambda x, \lambda y) \in \mathcal{R}$  the oriented angle  $(V, V_\lambda)$  is positive.

(ii) If for some times  $t_0$  and  $t_1$ ,  $\lambda M(t_0) = M_\lambda(t_1)$  then for  $\tau > 0$  small,  $M_\lambda(t_1 + t)$  is above the straight line  $OM(t_0 + t)$  for all  $t \in (0, \tau]$ .

To see the second point, remark that the two associated vectors  $\lambda V$  and  $V_\lambda$  have the same abscissa  $\lambda F(x, y)$ , and  $G(\lambda x, \lambda y) > \lambda G(x, y)$ .

We start by the following lemma:

**Lemma 8.** *For  $\lambda > 1$ , the homothetic  $\lambda\mathcal{O}$  of the orbit  $\mathcal{O}$  is below the orbit  $\mathcal{O}_\lambda$  (in  $\mathcal{R}$ ).*

**Proof.** Let  $\delta \in (0, T(y_0))$  and let  $\mathcal{O}_{\lambda, \delta}$  denote the orbit through the point  $\lambda M(\delta)$ . Using the last remark (i) there exists  $\varepsilon > 0$  such that  $\lambda \mathcal{O}$  is strictly below the curve  $\mathcal{O}_{\lambda, \delta}$  for  $t$  in the interval  $]0, \varepsilon[$ .

Suppose that there exists a point of intersection between the two curves  $\mathcal{O}_{\lambda, \delta}$  and  $\lambda \mathcal{O}$  in the set  $R$  and let  $(\lambda x_1, \lambda y_1) \in \mathcal{O}_\lambda$  be the first point of intersection. Thus, at the point  $(\lambda x_1, \lambda y_1)$ , the oriented angle between  $V$  and  $V_\lambda$  is negative, which contradicts Remark 6(i). Therefore, the curve  $\mathcal{O}_{\lambda, \delta}$  is above  $\lambda \mathcal{O}$  for  $t > \delta$ . Letting  $\delta \rightarrow 0$  and using the continuity of the solutions with respect to the initial data in  $\mathbb{R}^2 \setminus \{(0, 0)\}$  implies that the  $y$ -coordinate of any points of  $\mathcal{O}_\lambda$  in  $\mathcal{R}$  is greater than or equal to the  $y$ -coordinate of the point in  $\lambda \mathcal{O}$  which has the same  $x$ -coordinate. Since equality is impossible, by Remark 6, the proof follows.  $\square$

Now we can show the main result of this section:

**Theorem 3.** *The function  $y \mapsto T(y)$  is increasing.*

**Proof.** By Remark 6(ii) and since  $\lambda M(0) = M_\lambda(0)$ , there exists  $t_0 > 0$  such that for all  $t \in ]0, t_0]$  the point  $M_\lambda(t)$  is above the straight line  $OM(t)$ .

Let  $\tau > 0$  such that,  $M_\lambda(t_0 + \tau)$ ,  $M(t_0)$  and the origin are collinear. Then there exists  $\lambda' > \lambda$  such that  $M_\lambda(t_0 + \tau) = \lambda' M(t_0)$ . By the same argument, there exists  $t_1 > t_0$ , such that, for all  $t \in [t_0, t_1]$ ,  $M_\lambda(t + \tau)$  is above the line  $OM(t)$ . This proves that for all  $t$ ,  $M_\lambda(t)$  is strictly above the straight line  $OM(t)$ .  $\square$

### 3.4. Proof of Theorem 1

By Proposition 1, we know that all the  $2\pi$ -periodic (non-constant) solutions of (1.4) are  $\pi/k$ -anti-periodic for some positive integer  $k$ . Therefore (and up to a  $U(1)$  action), the completion of the proof is reduced to proving the uniqueness holds for positive solutions on the same interval, and that it vanishes at the end points. Let  $\omega$  and  $\tilde{\omega}$  be two solutions of (1.4) on  $(0, \theta_k)$  such that

$$\omega(\theta) > 0, \tilde{\omega}(\theta) > 0, \quad \forall \theta \in (0, \theta_k) \quad \text{and} \quad \omega(0) = \tilde{\omega}(0) = \omega(\theta_k) = \tilde{\omega}(\theta_k) = 0. \quad (3.4)$$

If  $\omega$  and  $\tilde{\omega}$  differ, their derivatives at  $\theta = 0$  must be different (since the equation is never degenerate on a trajectory). Therefore, we can assume

$$0 < \omega'(0) < \tilde{\omega}'(0). \quad (3.5)$$

Moreover,  $\tilde{\omega}'(0) < b$  since  $\tilde{\omega}$  is a periodic solution. If we put

$$F(x, y) := y$$

and

$$G(x, y) := \frac{|x|^{q-1} x [\beta^2 x^2 + y^2]^{2-p/2} - a_{p,q} \beta^2 x^3 - (a_{p,q} + (p-2)\beta^2) x y^2}{(p-1)y^2 + \beta^2 x^2},$$

then  $F$  and  $G$  satisfies the regularity and equivariance properties required in (3.3). Moreover, for any  $\lambda$ ,

$$F(\lambda x, \lambda y) = \lambda F(x, y)$$

and

$$G(\lambda x, \lambda y) = (\lambda^{q+1-p} - \lambda) \frac{|x|^{q-1} x [\beta^2 x^2 + y^2]^{2-p/2}}{(p-1)y^2 + \beta^2 x^2} + \lambda G(x, y).$$

Thus, assumption (3.3) is satisfied in whole  $Q_+$  and the minimal periods of  $\omega$  and  $\tilde{\omega}$  cannot be the same.  $\square$

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