

Available online at www.sciencedirect.com

Ann. I. H. Poincaré – AN ●●● (●●●●) ●●●—●●●


 ANNALES
 DE L'INSTITUT
 HENRI
 POINCARÉ
 ANALYSE
 NON LINÉAIRE
www.elsevier.com/locate/anihpc

Blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions

M. Jazar^{a,*}, R. Kiwan^b^a *Mathematics department, Lebanese University, P.O. Box 826, Tripoli, Lebanon*^b *Laboratoire de Mathématiques et Physique Théorique, Université de Tours, UMR 6083 du CNRS, Parc de Grandmont, 37200 Tours, France*

Received 9 November 2006; accepted 4 December 2006

Abstract

In this paper we give a positive answer to the conjecture proposed in [A. El Soufi, M. Jazar, R. Monneau, A Gamma-convergence argument for the blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24 (1) (2007) 17–39] by El Soufi et al. concerning the finite time blow-up for solutions of the problem (1), (2) below. More precisely, we give a direct proof of [A. El Soufi, M. Jazar, R. Monneau, A Gamma-convergence argument for the blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24 (1) (2007) 17–39, Theorem 1.1] and the conjecture given for the case $p > 2$.

© 2007 Elsevier Masson SAS. All rights reserved.

Résumé

Dans cet article on donne une réponse positive à la conjecture proposée dans [A. El Soufi, M. Jazar, R. Monneau, A Gamma-convergence argument for the blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24 (1) (2007) 17–39] par El Soufi et al. concernant l'explosion en temps fini des solutions du problème (1), (2) ci-dessous. Plus précisément, on donne une preuve directe du [A. El Soufi, M. Jazar, R. Monneau, A Gamma-convergence argument for the blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24 (1) (2007) 17–39, Theorem 1.1] ainsi que la conjecture énoncée pour le cas $p > 2$.

© 2007 Elsevier Masson SAS. All rights reserved.

Consider a bounded regular (of class C^2) domain Ω in \mathbb{R}^N , and study the solutions $u(x, t)$ of the following equation for some $p > 1$ (denoting the mean value $\frac{1}{|\Omega|} \int_{\Omega} f$ by \bar{f}_{Ω} for a general function f):

$$\begin{cases} u_t - \Delta u = |u|^p - \bar{f}_{\Omega} |u|^p & \text{on } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

* Corresponding author.

E-mail addresses: mjazar@ul.edu.lb (M. Jazar), kiwan@lmpt.univ-tours.fr (R. Kiwan).¹ The first author is supported by a grant from Lebanese University.

with the initial condition

$$\begin{cases} u(x, 0) = u_0(x) & \text{on } \Omega, \\ \text{with } \int_{\Omega} u_0 = 0. \end{cases} \quad (2)$$

Without loss of generality, we may assume that the measure of Ω is 1: $|\Omega| = 1$ (see [1]).

In this short note we show the following theorem:

Theorem 1 (Sufficient condition for blow-up, $p > 1$). *Let $p > 1$ and let u be a solution of (1), (2) with $u_0 \in C(\overline{\Omega})$, $u_0 \not\equiv 0$. If the energy of u_0 ,*

$$E(u_0) := \int_{\Omega} \left[\frac{1}{2} |\nabla u_0|^2 - \frac{1}{p+1} |u_0|^{p+1} \right] dx$$

is non-positive, that is

$$E(u_0) \leq 0, \quad (3)$$

then the solution does not exist in $L^\infty((0, T); L^2(\Omega))$ for all $T > 0$.

We refer to [1] for the motivation and references concerning the study of Eq. (1) above. In particular, we have the following local existence result

Theorem 2 (Local existence result, $1 < p < +\infty$). [1, Theorem 2.2] *For every $u_0 \in C(\overline{\Omega})$ there is a $0 < t_{\max} \leq \infty$ such that the problem (1), (2) has a unique mild solution, i.e. a unique solution*

$$u \in C([0, t_{\max}); C(\overline{\Omega})) \cap C^1((0, t_{\max}); C(\overline{\Omega})) \cap C((0, t_{\max}); D)$$

of the following integral equation

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} f(u(s)) ds$$

on $[0, t_{\max})$, with

$$f(u(s)) = |u(s)|^p - \int_{\Omega} |u(s)|^p.$$

Moreover, we have

$$\int_{\Omega} u(t) = 0 \quad \text{for all } t \in [0, t_{\max}) \quad (4)$$

and if $t_{\max} < \infty$ then

$$\lim_{t \uparrow t_{\max}} \|u(t)\|_{L^\infty(\Omega)} = \infty.$$

Proof of Theorem 1. We start with the following

Lemma 1. *For all t we have*

$$E(u(t)) = E(u_0) - \int_0^t \int_{\Omega} u_t^2. \quad (5)$$

Proof. A direct computation yields

$$\frac{d}{dt} E(u(t)) = - \int_{\Omega} u_t (\Delta u + u^p) = - \int_{\Omega} u_t^2.$$

Thus, integrating from 0 to t one gets (5). \square

Lemma 2. Define the two functions

$$m(t) := \frac{1}{2} \int_{\Omega} u^2(t) dx \quad \text{and} \quad h(t) := \int_0^t m(s) ds.$$

Then the two functions satisfy

$$m'(t) \geq (p+1) \iint_{0,\Omega} u_t^2 dx, \tag{6}$$

$$m'(t) \geq \lambda(p-1)m(t), \tag{7}$$

$$\frac{p+1}{2} (h'(t) - h'(0))^2 \leq h(t)h''(t). \tag{8}$$

Proof. We have

$$\begin{aligned} m'(t) &= \int_{\Omega} u_t u \\ &= \int_{\Omega} u \left(\Delta u + |u|^p - \int |u|^p \right) \\ &= - \int_{\Omega} \nabla u^2 + \int_{\Omega} u |u|^p - \int_{\Omega} u \int_{\Omega} |u|^p, \\ m'(t) &= -(p+1)E(u) + \frac{p-1}{2} \int_{\Omega} \nabla u^2. \end{aligned} \tag{9}$$

From one side, this last equality implies that

$$m'(t) \geq -(p+1)E(u) = -(p+1)E(0) + (p+1) \iint_{0,\Omega} u_t^2$$

by using Lemma 1, and by the hypothesis (3) we get (6).

On the other side, (9), (3) and Poincaré's inequality gives

$$m'(t) \geq \left(\frac{p-1}{2} \right) \int_{\Omega} \nabla u^2 \geq \frac{\lambda(p-1)}{2} \int_{\Omega} u^2 = \lambda(p-1)m(t),$$

which is (7). Now,

$$\begin{aligned} h'(t) - h'(0) &= \int_0^t m'(s) ds = \iint_{0,\Omega} u u_t \\ &\leq \left(\iint_{0,\Omega} u^2 \right)^{1/2} \left(\iint_{0,\Omega} u_t^2 \right)^{1/2} \end{aligned}$$

$$\begin{aligned} &\leq \left(\frac{2}{p+1}\right)^{1/2} (h(t))^{1/2} (m'(t))^{1/2} \\ &= \left(\frac{2}{p+1}\right)^{1/2} (h(t))^{1/2} (h''(t))^{1/2}; \end{aligned}$$

but, by (6)

$$h'(t) - h'(0) = \int_0^t m'(s) ds \geq (p+1) \iint_{0\Omega} u_t^2 \geq 0.$$

Therefore

$$\frac{p+1}{2} (h'(t) - h'(0))^2 \leq h(t)h''(t). \quad \square$$

Proof of the theorem. Assume that the solution exists for all t , then (7) implies that

$$\lim_{t \rightarrow \infty} h'(t) = \lim_{t \rightarrow \infty} m(t) = +\infty. \quad (10)$$

Thus, for all $0 < B < p+1$, there exists T_B such that, for all $t \geq T_B$

$$Bh'(t)^2 \leq (p+1)[h'(t) - h'(0)]^2.$$

Hence, by (8), we have for all $t \geq T_B$

$$Bh'(t)^2 \leq 2h(t)h''(t).$$

Consider now the function $H(t) = [h(t)]^{-q}$ for some $q > 0$ to be determined later. We have

$$\begin{aligned} H''(t) &= qh(t)^{-(q+2)} [(q+1)h'^2(t) - h(t)h''(t)] \\ &\leq qh(t)^{-(q+2)} \left[\frac{2(q+1)}{B} - 1 \right] h(t)h''(t), \end{aligned}$$

for $t \geq T_B$. In order to choose H concave on $[T_B, \infty)$, take B and $q > 0$ such that $q < \frac{p-1}{2}$ and $2(q+1) < B < p+1$. Consequently, H is a positive concave function on $[T_B, +\infty[$ and, by (10), $\lim_{t \rightarrow \infty} H(t) = 0$, which is a contradiction. \square

References

- [1] A. El Soufi, M. Jazar, R. Monneau, A Gamma-convergence argument for the blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (1) (2007) 17–39.