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Blow-up results for some second-order hyperbolic inequalities with a nonlinear term with respect to the velocity

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Abstract

We give sufficient conditions on the initial data so that a semilinear wave inequality blows up in finite time. Our method is based on the study of an associated second-order differential inequality. The same method is applied to some semilinear systems of mixed type.

Keywords: Wave equations and systems; Hyperbolic–elliptic systems; Hyperbolic–parabolic systems; Ordinary differential equations and inequalities; Finite time blow-up; Asymptotic behavior of solutions; Invariant regions

1. Introduction

Let Ω be a bounded regular[2] domain of $\mathbb{R}^N$ and let us denote respectively by $|\cdot|$ and $(\cdot,\cdot)$ the natural norm and inner product of the space $H = L^2(\Omega)$. Let $V \hookrightarrow H$ be a real Hilbert space that is continuously and densely embedded into H. Denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle_V$ respectively the

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1 The author is supported by a research grant from the Lebanese University.
2 The regularity needed here is that ensuring the existence of a first eigenvalue with positive associated eigenfunction for L.

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norm and the inner product of V and by $L \in \mathcal{L}(V, V^*)$ the unique linear continuous operator from V to V^*, the dual of V, satisfying

$$\langle Lu, v\rangle_{V^*,V} = \langle u, v\rangle_V, \quad (u, v) \in V \times V.$$

Moreover we assume that the operator L admits an eigenvalue λ (that we may assume to be positive without loss of generality) with a positive associated eigenfunction $\varphi \in V$.

Consider the semilinear evolution second-order partial differential inequality

$$\begin{cases} u'' + Lu \geqslant g(u') & \text{in } \mathcal{D}'\big((0,T)\times\Omega\big),\\ u(0,x) = u_0(x) & \text{on } \Omega,\\ u_t(0,x) = u_1(x) & \text{on } \Omega, \end{cases} \tag{1}$$

on the phase space $(u_0, u_1) \in V \times H$.

The model problem is a semilinear wave equation with Dirichlet boundary conditions, that is $V = H_0^1(\Omega)$, $L = -\Delta$, see the book of S. Alinhac [1] for a short view on semilinear hyperbolic problems and the question of blow-up. The particular case of equation with $g(x) = -x|x|^r$ is a dissipative semilinear wave equation that has been extensively studied in the literature, and many results on the existence of global solutions as well as the decay estimates have been established. See for instance [3,4,9,17].

The case of equation with $g(x) = x|x|^r$ is more complicated. For instance, a local existence-uniqueness of solutions is only guaranteed for small r and more regular initial data. See for instance [4,6] for equations with nonlinearities on u and u_t. In [8], A. Haraux showed that this problem (with $g(x) = x|x|^x$) has no nontrivial global and bounded solutions [8, Proposition 1.2]. Moreover, he constructed examples of blowing-up solutions with arbitrarily small initial data [8, Proposition 1.4 and Corollary 1.5]. See also the work of H. Levine, S.R. Park and J. Serrin [12] and references therein for a study of the long-time behavior of solutions for equations with $f(u) - g(u_t)$ on the right-hand side of the equation on $\Omega = \mathbb{R}^N$.

In this paper we consider problem (1) where g is a positive convex function $C^1(\mathbb{R}, \mathbb{R}^+)$ satisfying the growth condition

$$g(x) \geqslant C|x|^q \quad \text{for all } x \in \mathbb{R}, \tag{2}$$

where $C > 0$ is a constant and $q > 1$ if the dimension $N = 1$ or 2 or $1 < q \leqslant \frac{N+2}{N-2}$ if $N \geqslant 3$. We provide sufficient conditions on the initial data (u_0, u_1) so that the solution of problem (1) blows up in finite time.

The method used here is the so-called "eigenfunction method" which leads to the study of a second-order ordinary differential inequality that we make using invariant regions. This is the purpose of Section 2. In Section 3 we apply this to the semilinear wave equation (1). S. Kaplan [10] was the first to use Fourier coefficient's method in this setting. Later, R. Glassey [7] applied this method to nonlinear wave equations. In particular he treated an equation similar to (1) in the whole space $\mathbb{R}^N$. H. Levine [11, Section 5] considered equation similar to (1) on bounded domain with Dirichlet boundary conditions. Under some hypotheses on the nonlinearity g, the eigenvalue λ and the initial data, Levine gives a necessary condition for finite time blow-up. Our assumptions on the nonlinearity are different, in particular g does not need to be nondecreasing. Moreover, the method we use for the study of the ODI is different from that given in [11]. However, the model case, the case of equality with $g(s) = C|s|^q$, is covered by Levine's work (see [11, Section 5]), but our assumptions on the initial data are weaker. In Section 3 (Remark 3) we give a comparison between Levine's result and ours.

In Section 4 we apply our result to semilinear wave systems of the form

$$\begin{cases} u_{tt} - \Delta u \geqslant |v_t|^p, \\ v_{tt} - \Delta v \geqslant |u_t|^q, \end{cases} \tag{3}$$

by means of an extension of results of Section 2 to an associated system of second-order ordinary differential inequalities. In the last section we obtain similar results for systems of mixed type: hyperbolic–elliptic and hyperbolic–parabolic systems (see for instance the work of Pohozaev and Véron [13]).

2. Finite time blow-up for a second-order ordinary differential inequality

Consider the ODI:

$$\begin{cases} v'' + av \geqslant bv'^q, \quad t \geqslant 0, \\ v(0) = v_0, \quad v'(0) = v_1, \end{cases} \tag{4}$$

where a, b are positive constants.

Notice that in [2,16] one can find a complete study of the equation $u'' + a|u|^{p-1}u = b|u'|^{q-1}u'$ where $a, b > 0$ and $p, q > 1$. In [14,15], Ph. Souplet proved the existence of some global solutions to Eq. (4).

Concerning the ODI (4) we distinguish two cases: $q \leqslant 2$ and $q > 2$.

Proposition 1. *Let* $q \leqslant 2$, $\alpha = \frac{2a}{bq}(\frac{4a}{b^2q})^{\frac{2-q}{2q-2}}$ *and* $v_0, v_1 \in \mathbb{R}$ *satisfying*

$$\left| \begin{array}{ll} 0 < v_1, & \\ av_0 + \alpha < \dfrac{b}{2}v_1^q & \text{if } v_1 < \left[\dfrac{4a}{b^2q}\right]^{\frac{1}{2q-2}}, \\ av_0 < \dfrac{b}{2}v_1^q & \text{if } v_1 \geqslant \left[\dfrac{4a}{b^2q}\right]^{\frac{1}{2q-2}}. \end{array} \right. \tag{5}$$

Then all C^2*-solutions of* (4) *blow up in finite time. Moreover,*

$$v'(t) \geqslant \left[v_1^{1-q} - \frac{q-1}{2}bt\right]^{\frac{1}{1-q}}. \tag{6}$$

The proof of the proposition is based on the idea that the inequality $v'' \geqslant bv'^q$ provides blowing-up solution in finite time $T < \frac{v_1^{1-q}}{b(q-1)}$. We look then for invariant regions under the ODI (4) in order to obtain $av^p \leqslant b\varepsilon v'^q$.

Proof of Proposition 1. We show first that the region $\mathcal{D}$ defined by (5) is invariant under (4). Denote by F the function defined over $\mathbb{R}$ by

$$F(x) := \begin{cases} 0 & \text{for } x \leqslant x_1 := -\frac{\alpha}{a}, \\ \left[\frac{2}{b}(ax+\alpha)\right]^{\frac{1}{q}} & \text{for } x_1 \leqslant x \leqslant 0, \\ \left[\frac{4a}{b^2q}\right]^{\frac{1}{2q-2}} & \text{for } 0 \leqslant x \leqslant x_2 := \frac{b}{2a}\left[\frac{4a}{b^2q}\right]^{\frac{q}{2q-2}}, \\ \left[\frac{2a}{b}x\right]^{\frac{1}{q}} & \text{for } x \geqslant x_2. \end{cases}$$

The region $\mathcal{D}$ is then defined by $[y > F(x)]$.

Let v be a solution of (4) satisfying (5). Setting $x = v$ and $y = v_t$, the ODI (4) can be transformed into the dynamical system:

$$\begin{cases} x' = P(x, y) := y, \\ y' = Q(x, y) \geqslant by^q - ax, \\ x(0) = v_0, \\ y(0) = v_1. \end{cases} \tag{7}$$

In order to prove that the region $\mathcal{D}$ is invariant we will show that the vector field defining the dynamical system is "entering" along the curve $y = F(x)$. This is clear on the semi-axis $(y'O)$ and on the other segment. On the arc $x > x_2$,

$$\frac{y'}{x'} \geqslant \frac{by^q - ax}{y} \geqslant \frac{b}{2} y^{q-1} \geqslant \frac{2a}{bqy^{q-1}} = F'(x).$$

Now, on the arc $x_1 < x < 0$ we have

$$\frac{y'}{x'} \geqslant \frac{\alpha}{y} \geqslant F'(x).$$

Therefore for all $t > 0$ we have $\frac{b}{2} v_t^q \geqslant av$, and then $v'' > \frac{1}{2} b v'^q$. Integrating we get (6). □

The following proposition gives a similar result for the case $q > 2$. We discuss later the difference between the corresponding invariant regions in these two cases.

Proposition 2. *Let $q > 2$ and $v_0, v_1 \in \mathbb{R}$ satisfying*

$$\left| \begin{array}{l} 0 < v_1, \\ bqv_1^{q-2}\big(bv_1^q - av_0\big) - a > 0. \end{array} \right. \tag{8}$$

Then all C^2-solutions of (4) *blow up in finite time. Moreover,*

$$v'(t) \geqslant \big[v_1^{1-q} - (q-1)(1-\varepsilon)bt\big]^{\frac{1}{1-q}}, \tag{9}$$

where $\varepsilon \in (0, 1)$ is a constant satisfying

$$\varepsilon bqv_1^{q-2}\big(\varepsilon bv_1^q - av_0\big) - a \geqslant 0. \tag{10}$$

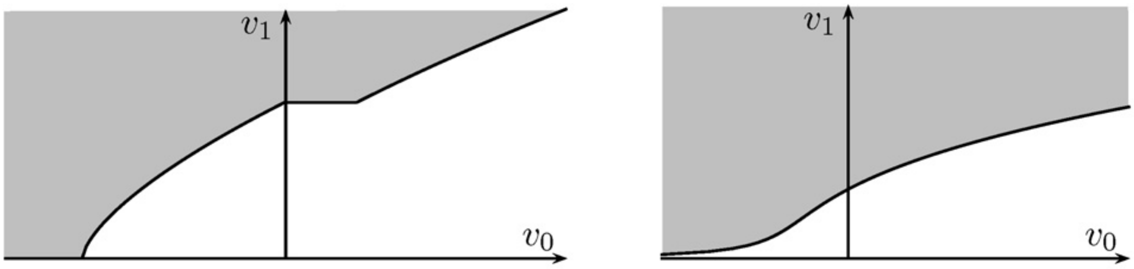
Proof. First note that, using (8) and taking

$$f(\varepsilon) = \varepsilon bqv_1^{q-2}\big(\varepsilon bv_1^q - av_0\big) - a,$$

there exists at least $\varepsilon \in (0, 1)$ satisfying $f(\varepsilon) > 0$.

Setting $A = \varepsilon bv_1^q - av_0$, we have $v_1^{q-2} \geqslant \frac{a}{\varepsilon bqA}$. Denote by $\mathcal{D}'$ the domain $[y > F_2(x)]$, where $F_2(x) := [\frac{1}{\varepsilon b(ax+A)}]^{1/q}$. Since y' remains positive, as long as (x, y) belongs to $\mathcal{D}'$, then for all $t > 0$, $(y(t))^{q-2} \geqslant \frac{a}{\varepsilon bqA}$. Following the same arguments as in the last proposition, we have, along the curve $y = F_2(x)$,

$$\frac{y'}{x'} \geqslant \frac{by^q - ax}{y} \geqslant \frac{\varepsilon by^q - ax}{y} = \frac{A}{y} \geqslant \frac{a}{\varepsilon bqy^{q-1}} = F_2'(x). \qquad \square$$



$a = 1,\ b = 2$ *and* $q = 1.5.$ $a = b = 1$ *and* $q = 2.5.$

Fig. 1.

Remark 1 *(On the admissible initial data region).* In Fig. 1 we draw the curve limiting the region for two values of q. The admissible region in each case is colored in gray.

Remark 2. The same method works for the differential inequality

$$v'' + av^p \geqslant bv'^q,$$

where $p \leqslant 1$.

3. Blow-up criteria for a semilinear wave inequality

In this section we assume that we have a Sobolev injection type theorem: $V \hookrightarrow L^{2^*}(\Omega)$, with $2^* = \frac{2N}{N-2}$. Consider the problem

$$\begin{cases} u \in C\big([0,T);V\big) \cap C^1\big([0,T);H\big) \cap C^2\big([0,T);V^*\big), \\ u'' + Lu \geqslant g(u') \quad \text{in } \mathcal{D}'\big((0,T)\times\Omega\big), \\ u(0,x) = u_0(x) \quad \text{on } \Omega, \\ u_t(0,x) = u_1(x) \quad \text{on } \Omega, \end{cases} \tag{11}$$

where g is a positive convex function $C^1(\mathbb{R},\mathbb{R}^+)$ satisfying the growth condition (2), $q > 1$ if the dimension $N = 1$ or 2 or $1 < q \leqslant \frac{N+2}{N-2}$ if $N \geqslant 3$ and $(u_0,u_1) \in V \times H$. Indeed, local existence is guaranteed under hypotheses of this type on both the power q and the smoothness of the initial data, see for instance [5,8]. Under such hypotheses, local existence can be easily obtained by classical fixed point argument and abstract semi-group theory.

Let λ be the first positive eigenvalue of the operator L on H, and φ a nonnegative associated eigenfunction satisfying $\int_\Omega \varphi = 1$. Denote also by $v_i := \int_\Omega u_i\varphi$, $i = 0, 1$. As for the ODI (4), we distinguish two cases depending on the value of q.

Theorem 1. *Let* $(u_0,u_1) \in V \times H$ *such that* v_0 *and* v_1 *satisfy*:

$$\text{if } q \leqslant 2 \quad \left| \begin{array}{ll} 0 < v_1, & \\ \lambda v_0 + \alpha < \dfrac{C}{2} v_1^q & \text{if } v_1 < \left(\dfrac{4\lambda}{C^2 q}\right)^{\frac{1}{2q-2}}, \\ \lambda v_0 \leqslant \dfrac{C}{2} v_1^q & \text{if } v_1 \geqslant \left(\dfrac{4\lambda}{C^2 q}\right)^{\frac{1}{2q-2}}, \end{array} \right.$$

and

$$\text{if } q > 2 \quad \left| \begin{array}{l} 0 < v_1, \\ Cqv_1^{q-2}\big(Cv_1^q - \lambda v_0\big) - \lambda > 0, \end{array} \right.$$

where $\alpha := \frac{2\lambda}{Cq}(\frac{4\lambda}{C^2 q})^{\frac{2-q}{2q-2}}$. *Then every solution of* (11) *blow up in finite time*

$$T_{\max} \leqslant T^* := \frac{v_1^{1-q}}{(1-\varepsilon)C(q-1)},$$

where $\varepsilon := \frac{1}{2}$ *if* $q \leqslant 2$, *and if* $q > 2$, ε *satisfies* $\varepsilon Cqv_1^{q-2}(\epsilon Cv_1^q - \lambda v_0) - \lambda \geqslant 0$. *Moreover,*

$$\big\|u_t(t)\big\|_{L^1(\Omega)} \geqslant \|\varphi\|_{L^\infty(\Omega)}^{-1}\big[v_1^{1-q} - (q-1)(1-\varepsilon)Ct\big]^{\frac{1}{1-q}}.$$

Proof. If u is a solution of (11), set $v(t) := \int_\Omega u(t)\varphi\,dx$. By the elliptic regularity theory $\varphi \in V$ and then v is twice differentiable with $v' = \int u_t\varphi$ and $v'' = \langle u_{tt}\varphi\rangle_{V^*,V}$ that we write also as $\int u_t t\varphi\,dx$.

Multiply Eq. (11) by $\varphi \geqslant 0$ and then integrate over Ω. We get after integrating by parts

$$\int_\Omega u_{tt}\varphi\,dx - \int_\Omega Lu\varphi\,dx \geqslant C\int_\Omega |u_t|^q\varphi\,dx,$$

$$v''(t) + \lambda v(t) \geqslant C\big|v'(t)\big|^q,$$

where we used Jensen's lemma. Using Propositions 1 and 2 we deduce that v blows up before the time T^* and

$$\|\varphi\|_{L^\infty(\Omega)}\big\|u_t(t)\big\|_{L^1(\Omega)} \geqslant \left|\int_\Omega u_t(t)\varphi\,dx\right| = \big|v'(t)\big| \geqslant \big[v_1^{1-q} - C(q-1)(1-\epsilon)t\big]^{\frac{-1}{q-1}}. \qquad \square$$

Remark 3 *(Comparison with the results of Levine [11]).* In [11, Theorem 5.1], Levine shows a blow-up result for a class of equations. Levine's result can be easily applied to Eq. (11) with $g(s) = C|s|^q$. However, the assumptions he needs on the initial data are not the same. Indeed, by [11, Hypothesis (5.5)], v_0 and v_1 satisfy

$$v_1 > v_0 > s_0$$

for some s_0 (s_0 should be in this case greater than $(\frac{\lambda+1}{C})^{\frac{1}{q-1}}$). This gives a region of the form shown in Fig. 2 to be compared with the regions of Remark 1.

Remark 4. According to Remark 2, the same method could be generalized to inequalities of the form ($p \leqslant 1$)

$$u_{tt} - \Delta u^p \geqslant g(u_t).$$

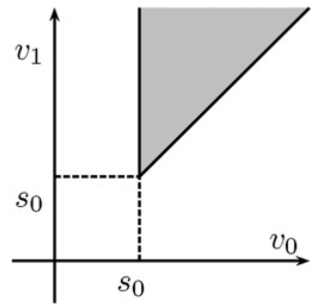


Fig. 2.

4. Application to systems of semilinear wave inequalities

In this section we show how to extend the same method to the following system

$$\begin{cases} u, v \in C\big([0,T); V\big) \cap C^1\big([0,T); H\big) \cap C^2\big([0,T); H^{-1}(\Omega)\big), \\ \left.\begin{aligned} u'' + Lu \geqslant |v'|^p \\ v'' + Lv \geqslant |u'|^q \end{aligned}\right\} \quad \text{in } \mathcal{D}'\big((0,T)\times\Omega\big), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \quad \text{on } \Omega, \\ v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x) \quad \text{on } \Omega, \end{cases} \tag{12}$$

where (u_0, u_1) and (v_0, v_1) are in $V \times H$ and the powers p and q are greater than 1 if the dimension $N = 1$ or 2 or in $(1, \frac{N+2}{N-2}]$ if $N \geqslant 3$. For simplicity we choose a simple form of the nonlinearity, although a general form is possible. For this, we introduce the following system of ODI:

$$\begin{cases} U'' + aU \geqslant V'^p, \quad t \geqslant 0, \\ V'' + aV \geqslant U'^q, \quad t \geqslant 0, \\ U(0) = U_0, \quad U'(0) = U_1, \\ V(0) = V_0, \quad V'(0) = V_1. \end{cases} \tag{13}$$

The method of Section 2 could be easily extended to the previous system as follows:

Proposition 3. *Let $1 < p \leqslant q$, U_0, V_0, U_1 and V_1 satisfying*

$$\left| \begin{aligned} & U_1, V_1 > 1, \quad U_0 V_0 \geqslant U_1 V_1, \\ & U_1^q \geqslant \left[a + \frac{1}{p}\right] V_0, \\ & V_1^p \geqslant \left[a + \frac{1}{p}\right] U_0. \end{aligned} \right. \tag{14}$$

Then every C^2-solution (U, V) of (13) *blows up in finite time. Moreover, we have*

$$U'(t) + V'(t) \geqslant \left[(U_1 + V_1)^{1-p} - \frac{p-1}{1+ap} 2^{1-p} t\right]^{\frac{1}{1-p}}.$$

Proof. Setting $x = U$, $y = V'$, $z = V$ and $t = U'$, then the system (13) is transformed into the 4-d dynamical system

$$\begin{cases} x' = t, & x(0) = U_0, \\ t' \geqslant y^p - ax, & t(0) = U_1, \\ z' = y, & z(0) = V_0, \\ y' \geqslant t^q - az, & y(0) = V_1. \end{cases}$$

Set $\alpha := 1 + \frac{1}{ap}$ and $f_p(x) := H(x)(\alpha a x)^{1/p}$, where H is the Heaviside function. Denote by $\mathcal{D}_1 := [y \geqslant f_p(x)]$ and $\mathcal{D}_2 := [t \geqslant f_q(z)]$. The hypothesis (14) can then be read as $(U_0, U_1) \in \mathcal{D}_1$ and $(V_0, V_1) \in \mathcal{D}_2$.

First notice that as long as (x, y, z, t) remains in $\mathcal{D}_1 \times \mathcal{D}_2$, U'' and V'' remain positive, so for all $t > 0$,

$$t^{q-1} y^{p-1} \geqslant U_1^{q-1} V_1^{p-1} = \frac{U_1^q V_1^p}{U_1 V_1} \geqslant \alpha^2 a^2 \frac{U_0 V_0}{U_1 V_1} \geqslant \alpha^2 a^2.$$

In order to show that (x, y, z, t) could not exit the domain $\mathcal{D}_1 \times \mathcal{D}_2$ let us examine the vector field along the boundary. On $[t = f_q(z)]$ we have

$$\frac{y'}{x'} \geqslant \frac{t^q - az}{t} = \frac{(\alpha - 1)az}{t} = \frac{\alpha - 1}{\alpha} t^{q-1} \geqslant \frac{\alpha - 1}{\alpha} \frac{\alpha^2 a^2}{y^{p-1}} = ap(\alpha - 1) f_p'(x) = f_p'(x).$$

A similar calculation holds for the other boundary.

Finally, setting $\beta := \frac{\alpha - 1}{\alpha}$, we have

$$\begin{cases} U'' \geqslant \beta V'^p, \\ V'' \geqslant \beta U'^q. \end{cases}$$

Adding these two equations, using the fact that $U' \geqslant 1$ and $V' \geqslant 1$ for all $t > 0$, we get

$$\frac{1}{2\beta} W'' \geqslant \frac{1}{2}\left(V'^p + U'^q\right) \geqslant \frac{1}{2}\left(V'^p + U'^p\right) \geqslant 2^{-p} W'^p, \tag{15}$$

where $W := U + V$. Integrating we obtain

$$W'(t) \geqslant \left(W_1^{1-p} - (p-1)\beta 2^{1-p} t\right)^{\frac{1}{1-p}}. \qquad \square$$

In order to apply the last result to the system (12), denote by $U_i := \int_\Omega u_i \varphi$ and $g_i := \int_\Omega v_i \varphi$, $i = 0, 1$. Using the same method as in the proof of Theorem 1 and applying Proposition 3 we get directly:

Theorem 2. *Let* $1 < p \leqslant q$, (u_0, u_1) *and* (v_0, v_1) *in* $V \times H$ *such that* U_0, V_0, U_1 *and* V_1 *satisfy the hypothesis*

$$\left| \begin{array}{l} U_1, V_1 > 1, \quad U_0 V_0 \geqslant U_1 V_1, \\ U_1^q \geqslant \left[\lambda + \dfrac{1}{p}\right] V_0, \\ V_1^p \geqslant \left[\lambda + \dfrac{1}{p}\right] U_0. \end{array} \right. \tag{16}$$

Then every solution of (12) *blows up in finite time. Moreover, we have*

$$\|u_t + v_t\|_{L^1(\Omega)} \geqslant \|\varphi\|_{L^\infty(\Omega)}^{-1} \left[(U_1 + V_1)^{1-p} - \frac{p-1}{1 + \lambda p} 2^{1-p} t\right]^{\frac{1}{1-p}}.$$

5. Application to systems of mixed types

In this section we show how to apply the result of Section 3 to some hyperbolic–elliptic and hyperbolic–parabolic systems.

5.1. Hyperbolic–elliptic system

Consider the following system

$$\begin{cases} u, v \in C\big([0,T]; H_0^1(\Omega)\big) \cap C^1\big([0,T]; L^2(\Omega)\big) \cap C^2\big([0,T]; H^{-1}(\Omega)\big), \\ \left.\begin{array}{l} u_{tt} - \Delta u \geqslant |v_t|^q \\ -\Delta v = u \end{array}\right\} \qquad \text{in } \mathcal{D}'\big((0,T)\times\Omega\big), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \quad \text{on } \Omega, \\ v(0,x) = v_0(x) \qquad \text{on } \Omega, \end{cases} \tag{17}$$

where $q > 1$.

Theorem 3. *Let $u_0, v_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$ such that $U_0 := \int_\Omega u_0 \varphi\, dx$ and $U_1 := \int_\Omega u_1 \varphi\, dx$ satisfy*:

$$\text{if } q \leqslant 2 \quad \left| \begin{array}{ll} 0 < U_1, & \\ \lambda U_0 + \alpha < \dfrac{\lambda^{-q}}{2} U_1^q & \text{if } U_1 < \left(\dfrac{4\lambda^{2q+1}}{q}\right)^{\frac{1}{2q-2}}, \\ \lambda U_0 \leqslant \dfrac{\lambda^{-q}}{2} U_1^q & \text{if } U_1 \geqslant \left(\dfrac{4\lambda^{2q+1}}{q}\right)^{\frac{1}{2q-2}}, \end{array}\right.$$

and

$$\text{if } q > 2 \quad \left| \begin{array}{l} 0 < U_1, \\ q\lambda^{-q} U_1^{q-2}\big(\lambda^{-q} U_1^q - \lambda U_0\big) - \lambda > 0, \end{array}\right.$$

where $\alpha := \frac{2\lambda^{1+q}}{q} (\frac{4\lambda^{2q+1}}{q})^{\frac{2-q}{2q-2}}$. Then every solution of (17) *blows up in finite time*

$$T_{\max} \leqslant T^* := \frac{U_1^{1-q} \lambda^q}{(1-\varepsilon)(q-1)},$$

where $\varepsilon := \frac{1}{2}$ if $q \leqslant 2$, and if $q > 2$, ε satisfies $\varepsilon C q v_1^{q-2}(\epsilon C v_1^q - \lambda v_0) - \lambda \geqslant 0$. Moreover,

$$\big\|u_t(t)\big\|_{L^1(\Omega)} \geqslant \|\varphi\|_{L^\infty(\Omega)}^{-1} \big[U_1^{1-q} - (q-1)(1-\varepsilon)\lambda^{-q} t\big]^{\frac{1}{1-q}}.$$

Proof. Denote by $U := \int_\Omega u\varphi\, dx$ and $V := \int_\Omega v\varphi\, dx$. Differentiating the second equation of (17), multiplying by φ, then integrating over Ω we get

$$\lambda V'(t) = U'(t). \tag{18}$$

Multiplying the first equation of (17) by φ and then integrating over Ω we get, after using (18),

$$U''(t) + \lambda U(t) \geqslant \lambda^{-q} \big|U'(t)\big|^q.$$

We conclude applying Propositions 1 and 2 with $a = \lambda$ and $b = \lambda^{-q}$. □

5.2. Hyperbolic–parabolic system

Consider the following system

$$\begin{cases} u, v \in C([0,T); H_0^1(\Omega)) \cap C^1([0,T); L^2(\Omega)) \cap C^2([0,T); H^{-1}(\Omega)), \\ \left.\begin{array}{l} u_{tt} - \Delta u \geqslant |v_t|^q \\ (u-v)_t - \Delta(u-v)^m \leqslant \beta(u-v)^p \end{array}\right\} & \text{in } \mathcal{D}'((0,T)\times\Omega), \\ u - v \geqslant 0 & \text{on } [0,T)\times\Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) & \text{on } \Omega, \\ v(0,x) = v_0(x) & \text{on } \Omega, \end{cases} \tag{19}$$

where $q > 1$, $m \geqslant 1 \geqslant p$ and $\beta \in \mathbb{R}$.

Theorem 4. *Let $u_0, v_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. Set $U_0 := \int_\Omega u_0\varphi\,dx$, $V_0 := \int_\Omega v_0\varphi\,dx$ and $U_1 := \int_\Omega u_1\varphi\,dx$. Assume that $\beta \leqslant 0$ or [$\beta \leqslant \lambda$ and $U_0 - V_0 > 1$], $u_0 - v_0 \geqslant 0$ and*

$$\text{if } q \leqslant 2 \quad \left| \begin{array}{ll} 0 < U_1, & \\ \lambda U_0 + \alpha < \frac{1}{2}U_1^q & \text{if } U_1 < \left(\frac{4\lambda}{q}\right)^{\frac{1}{2q-2}}, \\ \lambda U_0 \leqslant \frac{1}{2}U_1^q & \text{if } U_1 \geqslant \left(\frac{4\lambda}{q}\right)^{\frac{1}{2q-2}}, \end{array}\right. \tag{20}$$

and

$$\text{if } q > 2 \quad \left| \begin{array}{l} 0 < U_1, \\ qU_1^{q-2}(U_1^q - \lambda U_0) - \lambda > 0, \end{array}\right.$$

where $\alpha = \frac{2\lambda}{q}(\frac{4a}{q})^{\frac{2-q}{2q-2}}$.

Then every solution of (19) *blows up in finite time*

$$T_{\max} \leqslant T^* := \frac{U_1^{1-q}}{(1-\varepsilon)(q-1)},$$

where $\varepsilon := \frac{1}{2}$ if $q \leqslant 2$, and if $q > 2$ ε satisfies $\varepsilon q v_1^{q-2}(\varepsilon v_1^q - \lambda v_0) - \lambda \geqslant 0$. Moreover,

$$\|v_t(t)\|_{L^1(\Omega)} \geqslant \|u_t(t)\|_{L^1(\Omega)} \geqslant \|\varphi\|_{L^\infty(\Omega)}^{-1}\left[U_1^{1-q} - (q-1)(1-\varepsilon)t\right]^{\frac{1}{1-q}}.$$

Proof. Denote by $U := \int_\Omega u\varphi\,dx$ and $V := \int_\Omega v\varphi\,dx$. Multiplying the second equation of (19) by φ and then integrating over Ω we get after using Jensen's lemma

$$(U-V)' \leqslant \int_\Omega \left[\beta(u-v)^p - \lambda(u-v)^m\right]\varphi \leqslant \beta(U-V)^p - \lambda(U-V)^m. \tag{21}$$

By the hypothesis on β,

$$T_0 := \max\{0 \leqslant t < T_{\max} \text{ s.t. } U(s) - V(s) > 1\ \forall s \in [0,t]\}$$

is positive, and for all $t \in (0, T_0)$ we have $U' \leqslant V'$. Set

$$T := \max\{0 \leqslant t < T_0 \text{ s.t. } U'(s) \geqslant 0\ \forall s \in [0,t]\}.$$

Since $U_1 > 0$, one has $T > 0$. For all $t \in [0, T)$ we have $0 \leqslant U' \leqslant V'$ and, hence, $0 \leqslant U'^q \leqslant V'^q$. Multiplying the first equation of (19) by φ then integrating over Ω we get

$$U'' + \lambda U \geqslant U'^q \tag{22}$$

for all $t \in [0, T)$. In order to apply Propositions 1 and 2 we need to prove that T is large enough. Indeed, by the hypothesis (20), (U, U') remains in the invariant region defined by (20) (see the proofs of Propositions 1 and 2). Thus $U'(t) > 0$ for all t and we obtain the result by applying Propositions 1 and 2. □

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