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Regularized Scalar Operators

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Abstract—We introduce a C-regularized scalar operator. These have the properties of spectral operators of scalar type, except that the spectral measure is bounded and countably additive only after applying the regularizing operator C. We discuss the relationship between regularized scalar operators, regularized functional calculi, and generating a polynomially bounded regularized group.

Keywords—Functional calculus, Group of operators, Unbounded generalized scalar operators, Vector-valued measure.

1. DEFINITIONS AND THEOREMS

Scalar operators are a generalization, to arbitrary Banach spaces, of self-adjoint operators on a Hilbert space. An early disappointment was the realization that most standard differential operators are not scalar, unless the operator acts on a Hilbert space. There are at least two ways to remedy this inconvenience. The first alternative is to generalize the spectral measure to a spectral distribution defined on $\mathcal{D}(\mathbb{R})$, the space of test functions. This is done in [1], where there is given a generalization of Stone's theorem on a Banach space, and an extensive number of applications (see [2]). The second is to consider the spectral projections $E(\sigma)$ as unbounded operators and introduce an injective bounded operator C such that $E(\sigma)C$ is bounded for all $\sigma \in \mathcal{B}$, where $\mathcal{B}$ is the Borel sets of the real line. This provides uniform control over the unboundedness and gives tangible information about the semisimplicity manifold of Kantorovitz (see [3], and the reference therein; see Theorem 4).

Throughout this letter, $\mathcal{H}$ will be a Banach algebra of complex-valued functions, A is a linear operator with $\sigma(A) \subseteq \mathbb{R}$, $C \in \mathcal{L}(X)$ is injective, and $CA \subseteq AC$. We make $\mathrm{Im}\,(C)$, the image of C, into a continuously embedded Banach subspace of X, denoted by $[\mathrm{Im}(C)]$, with $\|x\|_{[\mathrm{Im}(C)]} = \|C^{-1}x\|$, for all $x \in [\mathrm{Im}(C)]$. We say that the complex number λ is in the *C-resolvent set*, $\rho_C(A)$,

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX if $(\lambda - A)$ is injective and $\mathrm{Im}(C) \subseteq \mathrm{Im}(\lambda - A)$. We say also, that the Banach algebra $\mathcal{H}$ is *admissible*, if $f_0(s) \equiv 1 \in \mathcal{H}$, and there exists λ such that $g_\lambda(s) \equiv (\lambda - s)^{-1} \in \mathcal{H}$.

DEFINITION 1. *Suppose $\mathcal{H}$ is an admissible Banach algebra. A C-regularized $\mathcal{H}$ functional calculus for A is a continuous map $\Lambda : \mathcal{H} \to \mathcal{L}(X)$ such that*

(1) $\Lambda(f_0) = C$;
(2) $\Lambda f \Lambda g = C\Lambda(fg)$, *for all* $f, g \in \mathcal{H}$; *and*
(3) *if* $g_r \in \mathcal{H}$, *then* $r \in \rho_C(A)$ *and* $\Lambda g_r = (r - A)^{-1}C$.

For $\mathcal{H} = H^\infty(\Omega)$, where Ω is an open subset of the complex plane, the following appears in [4].

THEOREM 2. *Suppose $\mathcal{H}$ is admissible. Then the following are equivalent.*

(a) *A has a C-regularized $\mathcal{H}$ functional calculus.*
(b) *There exists a Banach space Z such that $[Im(C)] \hookrightarrow Z \hookrightarrow X$, and $A|_Z$ has a $\mathcal{H}$ functional calculus such that $Cf(A|_Z)x = f(A|_Z)Cx$, for all $x \in Z$.*

If Λ is the C-regularized $\mathcal{H}$ functional calculus for A, and $f \mapsto f(A|_Z)$ is the $\mathcal{H}$ functional calculus for $A|_Z$, then $\Lambda f = f(A|_Z)C$, for all $f \in \mathcal{H}$.

DEFINITION 3. *We will say that A is C-regularized scalar if it has a C-regularized $B(\mathbb{R})$-functional calculus Λ, where $B(\mathbb{R})$ is the set of all bounded complex-valued Borel measurable functions on the real line, with the supremum norm, and there exists a family of operators $\{F(\sigma) \mid \sigma \in \mathcal{B}\}$ such that, for all $x \in X$, $F(\sigma)x$ defines a countably additive vector-valued measure, and for all $f \in B(\mathbb{R})$, $x \in X$, $\Lambda(f)x = \int_{\mathbb{R}} f(s)\, dF(s)x$.*

If $F(s) \equiv F((-\infty, s])$, then the family $\{F(s)\}_{s\in\mathbb{R}}$ is called a C-spectral family for A.

THEOREM 4. *The following are equivalent.*

(a) *A is C-regularized scalar.*
(b) *There exists a Banach space V such that*

$$[\mathrm{Im}(C)] \hookrightarrow V \hookrightarrow X,$$

and $A|_V$ is a scalar-type spectral operator that commutes with $C|_V$.

If E is the projection-valued measure for $A|_V$, and F as in Definition 3, then $F(\sigma) = E(\sigma)C$, for any $\sigma \in \mathcal{B}$.

THEOREM 5. *Suppose $\sigma(A) = \{a_k\}_{k=-\infty}^{\infty} \subseteq \mathbb{R}$, $\lim_{|k|\to\infty} |a_k| = \infty$, and A has a C-regularized $B(\sigma(A))$ functional calculus. Then for all $p > 0$, A is $(i + A)^{-p}C$-regularized scalar.*

DEFINITION 6. *A C-regularized semigroup is a strongly continuous family $\{W(t)\}_{t\in\mathbb{R}_+} \subseteq \mathcal{L}(X)$ such that*

(1) $W(0) = C$,
(2) $W(t)W(s) = CW(t+s)$, *for all* $s, t \in \mathbb{R}_+$.

The generator is defined by $Ax = C^{-1}\frac{d}{dt}W(t)x|_{t=0}$, with maximal domain, that is

$$\mathcal{D}(A) \equiv \left\{ x \in X \mid \lim_{t\to 0} \frac{1}{t}\left(W(t)x - Cx\right) \text{ exists and is in } \mathrm{Im}(C) \right\}.$$

$\{W(t)\}_{t\in\mathbb{R}}$ is called a C-regularized group if (1) and (2) hold, for all $s, t \in \mathbb{R}$. The generator is defined in the same way, except that in "$\lim_{t\to 0}$," t may be positive or negative.

In the case where the regularized group is temperate, we can show the following.

THEOREM 7. *iA generates an $(i - A)^{-k}$-regularized group $\{W(t)\}_{t\in\mathbb{R}}$ and there exists a polynomial R_ℓ, of degree ℓ, such that $\|W(t)\| \le |R_\ell(t)|$, for all t. Then A has an $(i-A)^{-(k+1)}$-regularized $W^{\ell+1,\infty}(\mathbb{R})$-functional calculus.*

2. THE PROOFS OF THE THEOREMS

PROOF OF THEOREM 2. For the implication (a)⇒(b), let us define, for any $f \in \mathcal{H}$, the possibly unbounded operator $f(A)$, with domain equal to $\mathcal{D}(f(A)) \equiv \{x \mid \Lambda f x \in \mathrm{Im}(C)\}$, by $f(A)x \equiv C^{-1}\Lambda f x$. Let $\mathcal{D} \equiv \bigcap_{f\in\mathcal{H}} \mathcal{D}(f(A))$, for $x \in \mathcal{D}, \|x\|_Z \equiv \sup\{\|f(A)x\| \mid f \in \mathcal{H}, \|f\|_{\mathcal{H}} \leq 1\}$, and we take $Z \equiv \{x \in \mathcal{D} \mid \|x\|_Z < \infty\}$. Choosing $f \equiv f_0$ implies that $Z \hookrightarrow X$.

Since, for any $y = Cx \in [\mathrm{Im}(C)]$,

$$\|y\|_Z = \sup\{\|(\Lambda f)x\| \mid f \in \mathcal{H}, \|f\|_{\mathcal{H}} \leq 1\} \leq \|\Lambda\| \, \|x\| \equiv \|\Lambda\| \, \|y\|_{[\mathrm{Im}(C)]},$$

hence, $[\mathrm{Im}(C)] \hookrightarrow Z$.

We claim that, for any $f \in \mathcal{H}$, $f(A)$ takes $\mathcal{D}$ to itself. Fix $x \in \mathcal{D}$, $g \in \mathcal{H}$. By definition of Λ, $C\Lambda g(f(A)x) = \Lambda g \Lambda f x = C\Lambda(fg)x$. The injectivity of C implies that $\Lambda g(f(A)x) = \Lambda(fg)x$. Since $x \in \mathcal{D} \subset \mathcal{D}(fg(A))$, thus we have $\Lambda g(f(A)x) \in \mathrm{Im}(C)$, that is, $f(A)x \in \mathcal{D}(g(A))$. Since g was an arbitrary element of $\mathcal{H}$, we infer that $f(A)x \in \mathcal{D}$. This argument also shows that $g(A)f(A)x = (fg)(A)x$, for all $x \in \mathcal{D}$, f, $g \in \mathcal{H}$.

If $\|g\|_{\mathcal{H}} = 1$, then $\|g(A)x\|_Z = \sup\{\|(fg)(A)x\| \mid f \in \mathcal{H}, \|f\|_{\mathcal{H}} \leq 1\} \leq \|x\|_Z$, because $\|fg\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}}\|g\|_{\mathcal{H}}$.

So $f(A) : Z \to Z$, with $\|f(A)x\|_Z \leq \|f\|_{\mathcal{H}}\|x\|_Z$. Thus, $A|_Z$ has a $\mathcal{H}$ functional calculus given by $f(A|_Z) \equiv (f(A))|_Z$.

Note that, for any $f \in \mathcal{H}$, $f(A)$ is closed, since $\Lambda f \in \mathcal{L}(X)$. By [5, Lemma 1.5], Z is a Banach space.

The implication (b)⇒(a) is clear by defining $\Lambda f \equiv Cf(A|_Z)$. ∎

PROOF OF THEOREM 4. Suppose (a), and let Z be the Banach space of Theorem 2 for $B(\mathbb{R})$. Let V be the set of all $x \in Z$, such that $\sigma \mapsto 1_\sigma(A|_Z)f(A|_Z)x$ defines a countably additive Z-valued measure, for any $f \in B(\mathbb{R})$. We give V the same norm as Z. For any $x \in V$, Borel set σ, define $E(\sigma)x \equiv 1_\sigma(A|_Z)x$.

It is clear that V is left invariant by $f(A|_Z)$, for any $f \in B(\mathbb{R})$; in particular, $E(\sigma)$ maps V into itself, for any Borel set σ. Thus, $A|_V$ has a $B(\mathbb{R})$-functional calculus defined by

$$f(A|_V) \equiv f(A|_Z)|_V.$$

We will show that V is a closed subspace of Z. Suppose $\{x_n\}_{n=1}^\infty$ is a sequence in V, converging to $x \in Z$. For any $f \in B(\mathbb{R})$, since $f(A|_Z) \in \mathcal{L}(Z)$, $f(A|_Z)x_n \to f(A|_Z)x$, in Z. Thus, to show countable additivity of $\sigma \to E(\sigma)f(A|_Z)x$, it is sufficient to assume that $f = f_0$, so that $f(A|_Z) = I$. Finite additivity follows from the fact that $g \mapsto g(A|_Z)$ is linear. Let $\{\sigma_k\}_{k=1}^\infty$ be a sequence of disjoint Borel sets, $\sigma \equiv \cup_{k=1}^\infty \sigma_k$. Since the map $g \mapsto g(A|_Z)$ is continuous, there exists a constant M such that $\|E(\Theta)\| \leq M$, for any Borel set Θ. Thus, a standard ϵ argument shows that

$$\lim_{N\to\infty} \left\| E(\sigma)x - \sum_{k=1}^{N} E(\sigma_k)x \right\| = \lim_{n\to\infty} \lim_{N\to\infty} \|E(\cup_{k=N+1}^\infty \sigma_k)x_n\| = 0,$$

since $\Theta \mapsto E(\Theta)x_n$ is countably additive, for all n. This shows that $x \in V$, so that V is a closed subspace of Z, hence, a Banach space.

Clearly E is a projection-valued measure and

$$f(A|_V)x = \int_{\mathbb{R}} f(s)\, dE(s)x, \qquad \text{for all } x \in V,$$

for any simple function f, thus for any $f \in C_0(\mathbb{R})$, since such a function is the uniform limit of simple functions. In particular,

$$(i + A|_V)^{-1}x = \left(t \mapsto (i+t)^{-1}\right)(A|_V)x = \int_{\mathbb{R}} (i+s)^{-1}\, dE(s)x, \qquad \text{for all } x \in V.$$

This implies that $A|_V$ is a scalar-type spectral operator.

It is clear from Theorem 2 that $F(\sigma) = E(\sigma)C$. This implies that $\mathrm{Im}(C) \subseteq V$, since it follows that $E(\sigma)f(A|_Z)Cx = \int_\sigma f(s)\, dF_x(s)$. By Theorem 2, $[\mathrm{Im}(C)] \hookrightarrow V$. The fact that (b)$\Rightarrow$(a) is clear by defining $F(\sigma) \equiv E(\sigma)C$, $\Lambda f \equiv Cf(A|_V)$ as in (2) of Definition 1. ∎

Proof of Theorem 5. Let Λ be the C-regularized $B(\sigma(A))$ functional calculus for A. Fix $p > 0$. For $f \in B(\sigma(A))$, define

$$\Lambda_p f \equiv (\Lambda f)\,(i + A)^{-p} = \Lambda f_p,$$

where $f_p(s) = f(s)(i+s)^{-p}$. It is clear that this defines a $C(i+A)^{-p}$-regularized $B(\sigma(A))$ functional calculus for A.

For any Borel set σ, define

$$F(\sigma) \equiv \Lambda_p\,(1_\sigma)\,.$$

Fix $x \in X$. For any $x^* \in X^*$, there exists a unique complex-valued measure of bounded variation μ_{x^*}, such that

$$\langle (\Lambda f)x, x^* \rangle = \sum_{k=-\infty}^{\infty} f(a_k)\mu_{x^*}(\{a_k\})\,, \tag{1}$$

for any $f \in C_0(\mathbb{R})$. In particular,

$$\langle F(\{a_k\})\,x, x^* \rangle = (i + a_k)^{-p}\,\mu_{x^*}(\{a_k\})\,, \qquad \text{for all } k \in \mathbb{Z}. \tag{2}$$

By the Dunford-Pettis theorem, the map $\sigma \mapsto F(\sigma)x$ is a countably additive vector-valued measure.

For $x^* \in X^*$, $f \in B(\sigma(A))$, (1) and (2) imply that

$$\langle (\Lambda_p f)\,x, x^* \rangle = \langle (\Lambda f_p)\,x, x^* \rangle = \sum_{k=-\infty}^{\infty} f(a_k)\,\langle F(\{a_k\})\,x, x^* \rangle = \int_{\mathbb{R}} f(s)\,\langle dF(s)x, x^* \rangle,$$

which implies $(\Lambda_p f)x = \int_{\mathbb{R}} f(s)\,dF(s)x$. ∎

Proof of Theorem 7. As in Definition 2.2, define $g_i(s) \equiv (i-s)^{-1}$.

Note that we think of $W(t)$ as being $(i-A)^{-k}e^{itA}$, so that, informally, we are using the Fourier inversion theorem,

$$\int_{\mathbb{R}} \mathcal{F}(fg_i)W(t)\,dt = (fg_i)(A)\,(i-A)^{-k} = f(A)\,(i-A)^{-(k+1)}\,.$$

Define, for any $x \in X$ and $f \in W^{\ell+1,\infty}(\mathbb{R})$,

$$\Lambda(f)x \equiv \int_{\mathbb{R}} [\mathcal{F}(fg_i)]\,(t)W(t)x\,dt,$$

where $\mathcal{F}(f)$ is the Fourier transform of f.

The following calculation shows that this integral is convergent, and provides the desired upper bound for $\|\Lambda(f)x\|$. For any $f \in W^{\ell+1,\infty}(\mathbb{R})$, there exist two constants K_1, K_2 such that

$$\begin{aligned}\|\Lambda(f)x\| &= \int_{\mathbb{R}} \left[\mathcal{F}\left(\left(1+\frac{d}{ds}\right)^{\ell}(fg_i)\right)\right](t)\frac{W(t)x}{(i+t)^{\ell}}\,dt \\ &\le K_1\|x\| \left\|\mathcal{F}\left(\left(1+\frac{d}{ds}\right)^{\ell}(fg_i)\right)\right\|_1 \\ &\le K_1K_2\|x\|\sum_{j=0}^{\ell+1}\left\|(fg_i)^{(j)}\right\|_2 \le K_1K_2M_{\ell+1}\|x\|\,\|f\|_{W^{\ell+1,\infty}(\mathbb{R})}.\end{aligned}$$

In fact, for all $f \in H^1(\mathbb{R})$, there exists a constant c so that $\|\mathcal{F}f\|_1 \le c(\|f\|_2 + \|f'\|_2)$, and for all $n \in \mathbb{N}$, there exists a constant M_n such that

$$\sum_{j=0}^{n} \left\| (fg_i)^{(j)} \right\|_2 \le M_n \|f\|_{W^{n,\infty}(\mathbb{R})}, \qquad \text{for all } f \in W^{n,\infty}(\mathbb{R}).$$

Thus, the map $f \mapsto \Lambda f$ defines a bounded linear map from $W^{\ell+1,\infty}(\mathbb{R})$ into $\mathcal{L}(X)$.

We will now verify (1)–(3) of Definition 1.

$$\begin{aligned}\Lambda(f_0)x &\equiv \int_{\mathbb{R}} \mathcal{F}(g_i)(t)W(t)x\,dt = -i\int_{-\infty}^{0} e^t W(t)x\,dt = -i\int_0^{\infty} e^{-t}W(-t)x\,dt \\ &= -i\,(1+iA)^{-1}\,W(0)x = (i-A)^{-1}\,W(0)x = (i-A)^{-(k+1)}\,x.\end{aligned}$$

This verifies (1). Property (3) follows by essentially the same calculation.

For (2), we use a change of variables and Fubini's theorem, as follows, for $x \in X$, f, $g \in W^{\ell+1,\infty}(\mathbb{R})$.

$$\begin{aligned}(\Lambda f \Lambda g)\,x &\equiv \int_{\mathbb{R}} [\mathcal{F}(fg_i)]\,(t_1)W(t_1)\left[\int_{\mathbb{R}} [\mathcal{F}(gg_i)]\,(t_2)W(t_2)x\,dt_2\right] dt_1 \\ &= \int_{\mathbb{R}} [\mathcal{F}(fg_i)]\,(t_1)\left[\int_{\mathbb{R}} [\mathcal{F}(gg_i)]\,(t_2)W\,(t_1+t_2)\,(i-A)^{-k}\,x\,dt_2\right] dt_1 \\ &= \int_{\mathbb{R}} W(t_3)\,(i-A)^{-k}\,x\left[\int_{\mathbb{R}} [\mathcal{F}(fg_i)]\,(t_1)\,[\mathcal{F}(gg_i)]\,(t_3-t_1)\;dt_1\right] dt_3 \\ &= \int_{\mathbb{R}} W(t_3)\,(i-A)^{-k}\,x\,[\mathcal{F}\,((fg_i)(gg_i))]\,(t_3)\,dt_3 \equiv \Lambda(fgg_i)\,(i-A)^{-k}\,x.\end{aligned}$$

This implies that

$$\Lambda(fg)\,(i-A)^{-(k+1)} = \Lambda(fg)\Lambda(f_0) = \Lambda(fgg_i)\,(i-A)^{-k},$$

thus,

$$\Lambda f \Lambda g = \Lambda(fg)\,(i-A)^{-(k+1)},$$

proving (2). ∎

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