



# Spectral Distributions and Some Cauchy Problems

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**Abstract**—In this paper we give a simple representation of the solution of the Cauchy problem when the operator admits a spectral distribution. First we apply this to the Schrödinger operator on  $\mathbb{R}^N$  with and without potential, and then on a bounded domain. In this case we give the expression of the associated spectral distribution. A second application is that for the Dirac operator. © 1999 Elsevier Science Ltd. All rights reserved.

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## 1. INTRODUCTION

A spectral distribution is an operator valued distribution defined on test functions (see [1,2]) that is extended naturally to  $C^\infty$  functions. The authors show that this symbolic calculus is admissible.

An important class of spectral distribution is that of order  $k$ , i.e., when the distribution could be extended continuously to the space  $\tilde{T}_k$ , space of functions  $f$  such that  $t^k \mathcal{F}f^{(k)} \in L^1(\mathbb{R})$ . A generalisation of Stone's theorem is then proved, which states that  $A$  generates a  $k$ -times integrated group  $(G(t))$  verifying  $\|G(t)\| \leq C|t|^k$  if and only if  $B = iA$  admits a spectral distribution of order  $k$ .

The motivation of semigroup theory is to solve the following problem:

$$\frac{d}{dt}u(t) = iBu(t), \quad u(0) = x. \quad (1)$$

When  $iB$  generates a  $C_0$ -semigroup  $(G(t))$ , the solution of (1) is then given by  $u(t) = G(t)x$ . In the case of integrated semigroup the expression is more complicated (see [3–5]). In this paper we will give a simple representation of the solution.

An extensive number of applications has been given in [6]. In particular they showed that the Schrödinger operator  $A = i\Delta$  in  $L^p(\mathbb{R}^N)$  with domain  $W^{2,p}(\mathbb{R}^N)$  verifies the condition of the generalized Stone's theorem and then  $iA$  admits a spectral distribution of order  $k \geq N|1/p - 1/2|$ .

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In bounded domain, recently, using Sobolev imbeddings and an interpolation result for integrated semigroups, Arendt [7] showed that the Schrödinger operator on  $L^p(\Omega)$ , where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ , generates a  $k$ -times integrated group for  $k \geq (N/2)|1/p - 1/2|$ . In Section 3 we give two ways to show that  $iA$  admits a spectral distribution of the same order  $k$ , the first based on Arendt's method, and the second is direct and explicit. Indeed we show how to construct the associated spectral distribution; nevertheless we lose in order of regularity.

A second application is given in Section 4 for the Dirac operator (see [8,9]). In [9], the author shows that the Dirac operator generates an  $\alpha$ -times integrated group for  $\alpha > 2|1/2 - 1/p| + 1$ . Here we show that it generates a  $k$ -times integrated group of type  $O(1 + |t|^k)$ ; hence it admits a  $\tilde{\mathcal{T}}_k$   $\mathcal{D}^{-k}$ -regularized spectral distribution for  $k \geq 2|1/2 - 1/p|$ .

## 2. SPECTRAL DISTRIBUTION AND THE CAUCHY PROBLEM

A *spectral distribution* is a linear algebra homomorphism  $\mathcal{E}$  from  $\mathcal{D} \equiv \mathcal{C}_0^\infty(\mathbb{R})$  into  $\mathcal{L}(X)$  verifying  $\lim_{n \rightarrow \infty} \mathcal{E}(\varphi_n) = I$ , where  $\varphi_n(t) \equiv \varphi(t/n)$ , with  $\varphi \in \mathcal{D}$  such that  $\varphi(0) = 1$ . For every  $\mathcal{C}^\infty$  function  $f$ ,  $\mathcal{E}(f)$  is defined as the limit of  $\mathcal{E}(\varphi_n f)$  as  $n \rightarrow \infty$ .

The spectral distribution  $\mathcal{E}$  is said to be of *degree*  $\ell$  if it can be extended as a linear continuous mapping on  $\tilde{\mathcal{T}}_\ell \equiv \mathcal{F}^{-1}\mathcal{T}_\ell$  equipped with the norm  $\Pi_\ell(f) \equiv p_\ell(\mathcal{F}f)$ , where  $\mathcal{F}$  is Fourier transformation and  $\mathcal{T}_\ell$  the completion of  $\mathcal{D}$  for the norm  $p_\ell$  defined by  $p_\ell(\varphi) \equiv \sum_{k=0}^\ell \|t^k \varphi^{(k)}\|_{L^1}$ .

LEMMA 2.1.

- (1) For all  $\varphi \in \mathcal{D}$ ,  $s > 0$ , define  $\varphi_s(t) \equiv s^{-1}\varphi(t/s)$ . Then for any positive integer  $k > 0$ ,  $p_k(\varphi_s)$  is independent of  $s$ . In the same way  $\Pi_k(\varphi_n)$  is independent of  $n$ , where  $\varphi_n(t) \equiv \varphi(t/n)$ .
- (2) For any integer  $\ell \geq 1$ ,  $\mu \in \mathbb{C}$ ,  $\operatorname{Re} \mu > 0$ , the function  $t \mapsto H(t)t^{\ell-1}e^{-\mu t}$  belongs to  $\mathcal{T}$ , where  $H(t)$  is the Heaviside function.
- (3) For any integer  $\ell \geq 1$ ,  $\lambda \in \mathbb{C}$  with  $\operatorname{Im} \lambda \neq 0$ , the function  $t \mapsto (\lambda - t)^{-\ell} \in \tilde{\mathcal{T}}$ .
- (4) For any integer  $\ell \geq 2$ ,  $\lambda \in \mathbb{C}$  with  $\operatorname{Im} \lambda \neq 0$ , and  $s \in \mathbb{R}$ , the function  $G_s : t \mapsto t(\lambda - t)^{-\ell} e^{ist}$  belongs to  $\tilde{\mathcal{T}}$ , and  $\Pi_k(G_s) \leq C(1 + |s|^k)$ .

PROOF.

- (1) Direct calculation.
- (2) For  $\ell = 1$  see [2]. If not, let  $m = \min\{k, \ell - 1\}$ , and we have

$$p_k(H(\tau)\tau^{\ell-1}e^{i\lambda\tau}) \leq \sum_{j=0}^m C_k^j \frac{(\ell-1)!}{(k-j-1)!} \|\mu^{k-j} H(\tau)\tau^{k+\ell-j-1} e^{-\mu t}\|_{L^1}$$

- (3) It suffices to notice that the Fourier transform of the function  $t \mapsto (\lambda - t)^{-\ell}$  is the function  $\tau \mapsto 1/\ell! H(\varepsilon\tau)(\varepsilon\tau)^{\ell-1} e^{i\lambda\varepsilon\tau}$ , where  $\varepsilon = +1$  if  $\operatorname{Im} \lambda > 0$  and  $\varepsilon = -1$  elsewhere.
- (4) Write

$$\frac{t}{(\lambda - t)^\ell} = \frac{\lambda}{(\lambda - t)^\ell} - \frac{1}{(\lambda - t)^{\ell-1}},$$

and for  $\ell - 1 \geq 1$  each one of them is in  $\tilde{\mathcal{T}}$ . Now note that if  $f$  verifies  $t^i \mathcal{F}f^{(k)} \in L^1$  for every  $i \leq k$ , then the function  $g : t \mapsto e^{ist} f(t)$  belongs to  $\tilde{\mathcal{T}}$  since  $\mathcal{F}g(\tau) = \mathcal{F}f(\tau - s/2\pi)$  and by a simple change of variable we get  $\Pi_k(g) = \|t^k \mathcal{F}f^{(k)}(t - s/2\pi)\|_{L^1} = \|(t + s/2\pi)^k \mathcal{F}f^{(k)}(t)\|_{L^1} \leq C(1 + |s|^k)$ . ■

**THEOREM 2.2.** *Let  $B$  be the momentum of a spectral distribution  $\mathcal{E}$  of degree  $k$ . For every  $x \in D(B^{k+1})$ ,  $s \in \mathbb{R}$ ,  $u(s) = \mathcal{E}(t \mapsto (\lambda - t)^{-k-1} e^{ist})(\lambda - B)^{k+1}x$  is the unique solution of the Cauchy problem (1), and this solution is polynomially bounded, i.e.,  $\|u(s)\| \leq C(1 + |s|^k)\|(\lambda - B)^{k+1}x\|$  for every  $s \in \mathbb{R}$ .*

Note that  $u(s)$  can be written in the form  $u(s) = (\lambda - B)^{-k-1}e^{isB}(\lambda - B)^{k+1}$  and this product is not commutative since  $e^{isB}$  is not a bounded operator (see [1, Section 4]).

PROOF. First of all we show that for every  $y \in X$ ,  $u = \mathcal{E}(t \mapsto (\lambda - t)^{-k-1}e^{ist})y \in D(B)$ . Writing  $B = \mathcal{E}(t)$  and by its definition, it is sufficient to show that  $\lim_{n \rightarrow \infty} \mathcal{E}(t\varphi_n)u$  exists. We have  $\mathcal{E}(t\varphi_n)u = \mathcal{E}(t\varphi_n)\mathcal{E}(t \mapsto (\lambda - t)^{-k-1}e^{ist})y = \mathcal{E}(\varphi_n)\mathcal{E}(t \mapsto t(\lambda - t)^{-k-1}e^{ist})y$ ; all these operators are bounded since these functions are in  $\tilde{\mathcal{T}}_k$  (see Lemma 2.1), so  $\lim \mathcal{E}(t\varphi_n)u$  exists and  $u \in D(B)$ .

Now for  $x \in X$  set  $y = (\lambda - B)^{k+1}x$  and  $u(s) = (\lambda - B)^{-k-1}e^{isB}(\lambda - B)^{k+1}$ ; by the last calculation and [1, Lemma 4.4] we have  $\frac{d}{ds}u(s, x) = \mathcal{E}(t \mapsto t(\lambda - t)^{-k-1}e^{ist})y = i\mathcal{E}(t)\mathcal{E}(t \mapsto (\lambda - t)^{-k-1}e^{ist})y = iBu(s, x)$  and  $u(0) = \mathcal{E}(t \mapsto (\lambda - t)^{-k-1})y = x$ . The growth inequality that verifies  $u$  is then a direct application of Lemma 2.1.(4).

For the uniqueness let  $v(s)$  be another solution of (1), then using Lemma 2.3, we have  $\frac{d}{ds}(\lambda - B)^{-k-1}e^{i(t-s)B}v(s) = -iB(\lambda - B)^{-k-1}e^{i(t-s)B}v(s) + (\lambda - B)^{-k-1}e^{i(t-s)B}\frac{d}{ds}v(s) = 0$ , so this function is constant and  $(\lambda - B)^{-k-1}v(t) = (\lambda - B)^{-k-1}e^{itB}v(0) = (\lambda - B)^{-k-1}e^{itB}x$ . Since  $(\lambda - B)^{-k-1}$  is injective we get the desired result,  $v(t) = u(t)$ . ■

### 3. SPECTRAL DISTRIBUTIONS AND INTERPOLATION

In [7], the authors give an interpolation theorem stating that each generator of integrated semigroup  $(S(t))$  generates a  $C_0$ -semigroup  $(U(t))$  on some subset  $G$  (see [7, Theorem 3.1 and Remark 3.2; 5]). In this section we will use those results to obtain spectral distributions.

Since the Schrödinger operator on  $L^2$  generates a unitary group, it is natural to consider the following, here we use notations of [7].

**PROPOSITION 3.1.** *If the  $C_0$ -semigroup  $\{U(t)\}$  is uniformly bounded on  $G$ , then the integrated semigroup  $\{S(t)\}$  is of type  $O(|t|^k)$ , i.e., satisfies  $\|S(t)\| \leq Ct^k$ , for all  $t \geq 0$ , where  $C$  is a positive constant.*

PROOF. Taking  $r \in \rho(A)$ ,  $x \in X$ ,  $y \equiv (r - A)^{-k}x \in D(A^k) = G$  (see [7, Remark 3.2]), we have

$$\begin{aligned} S(t)x &= (r - A)^k S(t)y \\ &= (r - A)^k \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} U(s)y \, ds \\ &= (r - A_G)^k \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} U(s)y \, ds \\ &= \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} (r - A_G)^k U(s)y \, ds, \end{aligned}$$

hence

$$\begin{aligned} \|S(t)x\| &\leq \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} C \|(r - A_G)^k U(s)y\| \, ds \\ &\leq \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} C \|U(s)y\|_G \, ds \\ &\leq \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} C \|y\|_G \, ds \\ &\leq C|t|^k \|y\|_G \\ &\leq M|t|^k \|x\|_E. \end{aligned}$$

■

Unfortunately, the converse is not true in general. However we have the following result.

**PROPOSITION 3.2.** *Let  $(G(t))_{t \geq 0}$  be a  $k$ -times integrated semigroup verifying  $\|G(t)\| \leq |p(t)|$ , where  $p$  is a polynomial of degree  $\leq k$ . Then  $G(t)$  verifies  $\|G(t)(\lambda - A)^{-k}\| \leq Ct^k$ , where  $C$  is a positive constant and  $\lambda \in \rho(A)$ .*

Note that this is natural since we have  $G^{(i)}(0) = 0$  for every  $i = 0, \dots, k-1$ .

PROOF. Note that we can replace  $|p(t)|$  by  $M(1+t^k)$ . Since it is obvious for  $t \geq 1$ , suppose that  $t < 1$ . For  $x \in D(A^k)$ , the application  $t \mapsto G(t)x$  is  $k$ -times differentiable vector valued function and

$$G(t)x = \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} G^{(k)}(s)x ds.$$

Since  $G^{(k)}(s)x = G(s)A^k x + \sum_{i=0}^{k-1} (s^i/i!)A^i x$ , we have  $\|G^{(k)}(s)x\| \leq K(1+s^k)\|x\|_{D(A^k)}$ , where  $\|x\|_{D(A^k)} = \sum_{i=0}^{k-1} \|A^i x\|$ . Therefore

$$\begin{aligned} \|G(t)x\| &\leq K\|x\|_{D(A^k)} \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} (1+s^k) ds \\ &\leq K\|x\|_{D(A^k)} \frac{t^k}{(k-1)!} (1+t^k) \leq Kt^k\|x\|_{D(A^k)}. \end{aligned}$$

The uniform boundedness principle gives us the desired inequality. ■

The following is a slight modification of [4, Theorem 2.4].

PROPOSITION 3.3. *Let  $A$  be the generator of a  $(\lambda - A)^{-k}$ -regularized semigroup  $\{W(t)\}$  that is of type  $O(1+t^k)$  for some  $\lambda$ . Suppose that  $0 \in \rho(A)$ , then  $A$  generates a  $k$ -times integrated  $\{S(t)\}$  semigroup of the same type.*

THEOREM 3.4. *Let  $A$  be as in Proposition 3.3. Then  $iA$  admits a  $\check{T}_k (w - H_p)^{-k}$ -regularized spectral distribution.*

PROOF. Exactly as in [1],  $\mathcal{E}(f) \equiv \int (-1)^k (\mathcal{F}f)^{(k)}(t)S(t) dt$ , where  $\{S(t)\}$  is as in the last proposition, define a regularized spectral distribution since, by Proposition 3.2,  $S(t)$  verifies  $\|S(t)(\lambda - A)^{-k}\| \leq Ct^k$ . ■

## 4. EXAMPLES

### 4.1. Schrödinger Operator

#### 4.1.A. Schrödinger operator on $L^p(\mathbb{R}^N)$

In [6], we showed that the Schrödinger operator without potential admits a spectral distribution for  $k > N|1/2 - 1/p|$ .

#### 4.1.B. Schrödinger operator on bounded domain

With the notation of Arendt, let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with smooth boundary, let  $\Delta_p$ , ( $1 \leq p \leq \infty$ ) be the Laplace operator on  $L^p(\Omega)$  with maximal domain,  $A_p \equiv i\Delta_p$ ,  $A_0$  and  $A_c$  are the parts of  $A_p$  on  $C_0(\Omega)$  (continuous functions vanishing at infinity) and  $C(\overline{\Omega})$ , respectively.

The following is a direct application of [7, Theorems 4.2 and 4.3; 1, Theorem 3.4], and Proposition 3.1.

COROLLARY 4.1. *Let  $\Omega \subset \mathbb{R}^N$  be a bounded open set of  $\mathbb{R}^N$ , with boundary of class  $C^2$ . Then we have the following.*

- (i) *If  $N \leq 3$ , then  $iA_p$  ( $1 \leq p \leq \infty$ ) as well as  $iA_0$  and  $iA_c$  admit a spectral distribution of order 1 in  $L^p(\Omega)$ ,  $C_0(\Omega)$ , and  $C(\overline{\Omega})$ , respectively.*
- (ii) *Let  $1 < p < \infty$ , if  $k \geq (N/2)|1/2 - 1/p|$ , then  $iA_p$  admits a spectral distribution of order  $k$  in  $L^p(\Omega)$ .*
- (iii) *If  $k > N/4$ , then  $iA_0$ ,  $iA_c$ ,  $iA_\infty$ , and  $iA_1$  admit a spectral distribution of order  $k$  in  $C_0(\Omega)$ ,  $C(\overline{\Omega})$ ,  $L^\infty(\Omega)$ , and  $L^1(\Omega)$ , respectively.*

REMARK 4.2. Using Theorem 2.4, we can translate the last corollary in terms of solution of the problem  $\frac{d}{dt}u(t) = Au(t)$ ,  $u(0) = x$ , to get a simple representation of the solution of the form  $u(t) = (\lambda - iA)^{-k-1}e^{tA}(\lambda - iA)^{k+1}x$  where  $A$  is one of the Schrödinger operators as in Corollary 4.1 on the mentioned domain with the associated order  $k$ , and the solution  $u(t)$  is polynomially bounded as in Theorem 2.2.

### Direct construction of the spectral distribution of the Schrödinger operator on bounded domain

Consider the Schrödinger equation  $\frac{\partial u}{\partial t} - i\Delta u = 0$ ,  $x \in \Omega$  with the initial condition  $u(0, x) = \phi(x)$ , where  $\Omega$  is a bounded open of  $\mathbb{R}^N$  of class  $C^2$ ,  $\phi \in L^p(\Omega)$ ,  $1 < p < \infty$ . Here we will give the expression of the spectral distribution generated by this equation.

We know that the spectrum of  $-\Delta$  is the set of an increasing sequence of positive eigenvalues  $(\lambda_n)$  which tends to infinity. Let  $(f_n)$  be the associated eigenvectors and let  $k$  be an integer such that  $k \geq (N/2)|1/2 - 1/p|$ . Now for every  $\psi \in \tilde{T}_k$  we define for  $\phi \in L^p(\Omega)$

$$\begin{aligned} \mathcal{E}(\psi)\phi(x) &= \sum_n c_n(\phi) f_n(x) \int_{\mathbb{R}} e^{-i\lambda_n t} (-it)^k \hat{\psi}(t) dt \\ &= \sum_n c_n(\phi) f_n(x) \psi^{(k)}(-\lambda_n), \end{aligned}$$

where  $\hat{\psi}$  is Fourier transform of  $\psi$ ,  $c_n(\phi)$  is the  $n^{\text{th}}$  Fourier coefficient,  $c_n(\phi) = \int \phi \bar{f}_n$ . Now since  $|c_n(\phi)| \leq \|\phi\|_{L^p} \|f_n\|_{L^q}$ , where  $q$  is the conjugate of  $p$ , we have

$$\|\mathcal{E}(\psi)\phi\|_{L^p} \leq \sum_n |c_n(\phi)| \|f_n\|_{L^p} \left| \psi^{(k)}(-\lambda_n) \right| \leq \|\phi\|_{L^p} \sum_n \|f_n\|_{L^p} \|f_n\|_{L^q} \left| \psi^{(k)}(-\lambda_n) \right|,$$

using Sobolev imbeddings, and for  $s \geq (N/2)|1/2 - 1/p|$ , we have  $\|f_n\|_{L^p} \leq C \|f_n\|_{H^{2s}} \leq C'(1 + \lambda_n^s) \|f_n\|_{L^2} = C'(1 + \lambda_n^s)$  and  $\|f_n\|_{L^q} \leq K \|f_n\|_{L^2} = K$ , where  $C$ ,  $C'$ , and  $K$  are constants. Hence we get

$$\begin{aligned} \|\mathcal{E}(\psi)\| &\leq M \sum_n \frac{1 + \lambda_n^s}{\lambda_n^k} \left| \lambda_n^k \psi^{(k)}(-\lambda_n) \right| \\ &\leq M \sum_n \frac{1 + \lambda_n^s}{\lambda_n^k} \left| \mathcal{F} \left( t^k \hat{\psi}^{(k)}(\lambda_n) \right) \right| \\ &\leq \pi_k(\psi) \sum_n \frac{1 + \lambda_n^s}{\lambda_n^k} \leq M'' \pi_k(\psi), \end{aligned}$$

since as  $n \rightarrow \infty$ ,  $\lambda_n \simeq n^{2/N}$ , hence  $(1 + \lambda_n^s)/(\lambda_n^k) \simeq (n^{2s/N})/(n^{2k/N})$ , which means that we must have  $2k/N > 2s/N + 1$ , i.e.,  $k$  must be  $k > (N/2)|1/2 - 1/p| + N/2$ .

Note that we have lose in order comparing with 4.1(ii).

#### 4.1.C. Schrödinger operator with potential

Denote by  $K^N$  the Kato class of measurable functions on  $\mathbb{R}^N$ , as defined in [10, p. 453]. In [11], it is shown that, for  $1 \leq p < \infty$ ,  $V_+ \in K^N$ ,  $V_- \in L^\infty(\mathbb{R}^N)$ ,  $iH_p \equiv i\Delta - iV$  generates an  $(w - H_p)^{-k}$ -regularized group on  $L^p(\mathbb{R}^N)$  that is of type  $O(1 + |t|^k)$  for any integer  $k > 2n|1/2 - 1/p|$ ; hence by Theorem 3.4,  $H_p$  admits a  $\tilde{T}_k (w - H_p)^{-k}$ -regularized spectral distribution.

#### 4.2. Dirac Operator

Let

$$\sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_j \equiv \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad (1 \leq j \leq 3),$$

then the Dirac operator is defined to be  $\mathcal{D} \equiv \mathcal{A} + \mathcal{B}$ , where

$$\mathcal{A} \equiv c \sum_{j=0}^n B_j \frac{\partial}{\partial x_j}, \quad \mathcal{B} \equiv \mu c^2 \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

$\mu$  is the mass of the electron,  $c$  is the speed of light.

In [9, Theorem 3.2], the author showed that  $\mathcal{D}$  defined on  $L^p(\mathbb{R}^3)^4$  ( $1 < p < \infty$ ) with domain  $D(\mathcal{A}) \times D(\mathcal{A})$  generates a  $k$ -times integrated semigroup whenever  $k > 2|1/2 - 1/p| + 1$ . In the following, Theorem 4.3, we will show that the Dirac operator generates a  $k$ -times integrated group of type  $O(1 + |t|^k)$  for  $k \geq 2|1/2 - 1/p|$ , hence  $i\mathcal{D}$  admits a  $\check{T}_k$   $\mathcal{D}^{-k}$ -regularized spectral distribution.

**THEOREM 4.3.** *The Dirac operator  $\mathcal{D}$  generates a  $k$ -times integrated group of type  $O(1 + |t|^k)$  for  $k \geq 2|1/2 - 1/p|$ , hence  $i\mathcal{D}$  admits a  $\check{T}_k$   $\mathcal{D}^{-k}$ -regularized spectral distribution.*

**PROOF.** Since  $i\mathcal{A}$  admits a spectral distribution of order  $k$  (see [12]), then  $\mathcal{A}$  generates  $\{W(t)\}$ , a  $(\lambda - \mathcal{A})^{-k}$ -regularized group of type  $O(1 + |t|^k)$ . And since  $\mathcal{B}$  generates a unitary group  $\{H(t)\}$ , then  $\mathcal{D}$  generates a  $(\lambda - \mathcal{D})^{-k}$ -regularized group of the same type. We get the desired result by applying Proposition 3.3.  $\blacksquare$

**REMARK 4.4.** In this case also, by using Theorem 2.4, we can translate the last corollary in terms of solution of the Cauchy problem as in 3.6  $u(t) = (\lambda - i\mathcal{D})^{-k-1} e^{t\mathcal{D}} (\lambda - i\mathcal{D})^{k+1} x$  where  $\mathcal{D}$  is the Dirac operator and the solution  $u(t)$  is polynomially bounded as in Theorem 2.4.

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