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Blow-up results for some nonlinear delay differential equations

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Abstract

In this work we study the blow up phenomena for some scalar delay differential equations. In particular, we make connection with the blow up of ordinary differential equations that are related to the delay differential equations.

1 Introduction

A great number of processes of the applied sciences can be modelled by means of delay differential equations: these are differential equations involving not only the function and their derivatives at time t but also the function (and their derivatives) at previous times. These equations has been subject of many studies. In this work, we intend to study the qualitative asymptotic behavior of such equations. The introduction of the delay in differential equations has many applications, for more details about this topics we refer to [1].

Differential inequalities arise when studying asymptotic behavior of a partial differential equation. In fact it is common to introduce an energy function and then transform the problem into the study of a differential inequality. This is also the case when studying asymptotic behavior of partial functional differential equations; for more details about this topic, we refer to [4]. For example, consider the reaction-diffusion parabolic equation

$$\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = -u|u|^{p-1}(t,x) + u(t-r,x),\ t \geq 0,\ x \in \Omega$$

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on a bounded regular domain Ω, with Dirichlet boundary conditions ($p > 1$). Setting $E(t) := \int_\Omega u^2(t,x)\,dx$, multiplying the equation by u and integrating over Ω, we get the following delay differential inequality:

$$2E'(t) \leq -\lambda_1 E(t) - CE^{\frac{p+1}{2}}(t) + E^{1/2}(t)E^{1/2}(t-r),$$

where λ_1 is the first eigenvalue of the Laplace operator and $C = C(\Omega)$ a positive constant. Setting $F(t) := \sqrt{E(t)}$, the last inequality becomes

$$4F'(t) \leq -\lambda_1 F(t) - CF^p(t) + F(t-r).$$

Using the comparison principle, we can see that $F(t) \leq G(t)$, where G is a solution of some equation of type (7) below (see Section 3). One can see that if G globally exists, then the same will hold for F and hence u globally exists in $L^2(\Omega)$. Meanwhile, if G blows-up in finite time, nothing can be said concerning F.

In this paper we focus on nonlinear functional differential equations with delay, and study global existence or finite time blow-up.
Consider the simple, but quite representative, ODE

$$\begin{cases} u'(t) = u^2(t), & t \geq 0 \\ u(0) = u_0. \end{cases}$$

For initial data $u_0 > 0$, it's clear that the unique solution exists in the time interval $[0, T^*[$, $T^* := 1/u_0$. One can see that the solution, given by $u(t) = 1/(T^* - t)$ is smooth on this interval, and that $\lim_{t \uparrow T^{*-}} u(t) = +\infty$. We say that the solution **blows-up in finite time** T^*, and the **rate** of this blow-up is $1/(T^* - t)$. It's direct that, by simple calculation, this is the case if we consider $u\prime(t) = u^p(t)$, $p > 1$, or more generally the equation

$$u'(t) = \theta(u(t)), \qquad t \geq 0,$$

with the initial data, $u(0) = u_0$, where θ is positive, continuous, and superlinear, or verifying the condition

$$\int_{u_0}^{\infty} \frac{ds}{\theta(s)} = T^* < \infty.$$

This Osgood's condition is necessary and sufficient for the occurrence of finite time blow-up for positive initial data (see [3] and the references therein).

Assume that we have information about the blow-up behavior of the equation

$$\begin{cases} z'(t) = f(z(t)), & t \geq 0 \\ z(0) = z_0, \end{cases} \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz. What happens if we add, for example, a new term that depends on a previous time. In this work, we will study the asymptotic behavior of some delay differential equations. In particular we study the effect of the delay term on the blow-up character: time of blow-up and rate. More precisely, we study the blow up of the following delay differential equations:

$$\begin{cases} y'(t) = f(y(t)) + g(y(t-r)), \ t \geq 0 \\ y_{|_{[-r,0]}} = \phi \in C := C\left([-r,0]\,;\mathbb{R}\right) \end{cases} \tag{2}$$

$$\begin{cases} y'(t) = f(y(t)) \times g(y(t-r)), \ t \geq 0 \\ y_{|_{[-r,0]}} = \phi \in C \end{cases} \tag{3}$$

or generally

$$\begin{cases} y'(t) = h(y(t), y(t-r)), \ t \geq 0 \\ y_{|_{[-r,0]}} = \phi \in C. \end{cases} \tag{4}$$

The phase space C is the space of continuous functions from $[-r, 0]$ to $\mathbb{R}$ provided with the uniform norm topology, $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions. The blow up phenomena is compared to the one of equation (1).

It is well known from [1], that if h is continuous, then for all $\phi \in C$, problem (4) has at least a solution which is defined on some maximal interval $[0, T_{\max}[$. Moreover if $T_{max} < \infty$, then the solution $y(t, \phi)$ blows up at T_{max}; this means that

$$\overline{\lim_{t \to T_{\max}}} |y(t, \phi)| = \infty. \tag{5}$$

The uniqueness of the solution holds if we assume that h verifies some Lipschitz conditions, for more details about this topic, we refer to [1].

The delay may change completely the dynamics of solutions. Consider for example, the following equation

$$y'(t) = y(t-2)y^2(t), \tag{6}$$

with initial data $y(\tau) = -\tau + 1$, for $\tau \in [-2, 0]$. Note that, on the interval $[0, 2]$, $y \geq y'^2$ and hence blows-up before the time $t = 1$.In fact the solution of (6) with it's initial data is given by $y(t) = \frac{2}{t^2 - 6t + 2}$, this blows-up at time $T := 3 - \sqrt{7} < 1$. While taking $y(\tau) = \tau - 1$ as initial data, then the solution $y(t) = \frac{-2}{t^2 - 2t + 2}$ exists for all t. No direct comparison could be easily made with some differential inequality.

In this work, we give sufficient conditions on the initial data ϕ, for which the associated solution exists globally or blow up in finite time.

In Section 3, we study the blow-up of the equation

$$\left\{\begin{array}{lcll} y'(t) & = & |y(t)|^p - |y(t-r)|^q & t>0 \\ y(\tau) & = & \phi(\tau) & \tau \in [-r,0] \end{array}\right. \tag{7}$$

where $p, q > 0$ and ϕ continuous on $[-r, 0]$. We will discuss the blow up phenomena in term of p, q and ϕ.

2 Blow-up phenomena for equation (2)

The aim of this section is to give simple conditions on the function g (or the function $y \longmapsto h(x, y)$ so that the blow-up phenomena of the equation with the delay term, equation (2), is the same as that of (1).

We start by the following fundamental and well known results:

Lemma 1 *(Comparison principle)[2]*
Let f_1, f_2 be two locally Lipschitz functions and u be a function of class C^1 on some interval $[a, b]$ verifying the differential inequality

$$f_1(u(t)) \leq u'(t) \leq f_2(u(t)), \qquad t \in [a, b].$$

Then for all $t \in [a, T[$ we have $u_1(t) \leq u(t) \leq u_2(t)$, where u_1, u_2 are solutions of

$$\left\{\begin{array}{ll} u_1'(t) = f_1(u_1(t)), & t \in [a, T_1] \\ u_2'(t) = f_2(u_2(t)), & t \in [a, T_2] \\ u_1(a) \leq u(a) \leq u_2(a); & \end{array}\right.$$

and $T := \min(T_1, T_2)$.

The proof of the following lemma is a consequence of the uniqueness property of problem (1).

Lemma 2 *Assume that f is locally Lipscitz. If $f(z_0) > 0$ then the solution z of the problem (1) is strictly increasing, if $f(z_0) < 0$ then z is strictly decreasing and if $f(z_0) = 0$ then z is constant.*

Concerning the first equation (2), we think that the delay term, the term $g(y(t-r))$ alone could not change the nature of existence. This seems to be direct in the case where the two terms have same nature: dissipative or source. Meanwhile, it is more complicated when the two terms have different nature.

Now we can give the first result.

Proposition 1 *Assume that*

(a) $xg(x) \geq 0$*, for all* x*.*

(b) For some $z_0 \geq 0$*, with* $f(z_0) > 0$*, the problem (1) blows-up in a finite time* $T_0 < \infty$*.*

Then for all $\phi \in C$*,* $\phi \geq 0$ *with* $\phi(0) \geq z_0$ *the problem (2) admits a maximal solution defined on an interval* $[-r, T_1[$ *with* $T_1 \leq T_0$*.*

Proof. First note that using Lemma 2, the solution of the problem (1) verifies $z(t) \geq 0$ for all t. By steps, we will show that $y(t) \geq z(t)$ for all t. Denoting by $T^* := \min(T_{max}, r)$, and since $\phi \geq 0$ and $xg(x) \geq 0$, we have $y'(t) \geq f(y(t))$. Since $y(0) = \phi(0) \geq z_0$, by the comparison principle we get $y(t) \geq z(t)$.
Repeating, if necessary, on each interval of length r, we get $y(t) \geq z(t) \geq 0$ for all $t \geq 0$, which gives the result. $\square$

Similarly we get the following

Proposition 2 *Assume that*

(a) $xg(x) \geq 0$*, for all* x*.*

(b) For some $z_0 \leq 0$ *with* $f(z_0) < 0$*, the problem (1) blows-up in a finite time* $T_0 < \infty$*.*

Then for all $\phi \in C$*,* $\phi \leq 0$ *with* $\phi(0) \leq z_0$*, the problem (2) admits a maximal solution defined on an interval* $[-r, T_1[$ *with* $T_1 \leq T_0$*.*

Remark. The maximal time of existence of $z' = z^2 + 1$, with $z(0) = 0$ is $T_0 = \pi/2$ while that of $y'(t) = y^2(t) + 1 + y(t - \pi/4)$ with $y(t) = 1$ on $[-\pi/4, 0]$ is $T_1 = \frac{\pi}{2\sqrt{2}} < T_0$.

Of course the condition "ϕ of constant sign" can be omitted if we suppose that g is positive.

Proposition 3 *Assume that*

(a) For all x, $g(x) \geq 0$.

(b) For some z_0 with $f(z_0) > 0$, the problem (1) blows-up in a finite time $T_0 < \infty$.

Then for all $\phi \in C$, with $\phi(0) \geq z_0$, the problem (2) blows-up in a finite time $T_1 \leq T_0$.

Proof. Following the same idea, $y'(t) \geq f(y(t))$, then by the comparison principle $y(t) \geq z(t)$ where z is the solution of (1). Next, notice that by Lemma 2, $z(t)$ increases to $+\infty$. $\square$

The general case, where the two functions f and g have different signs, is more complicated. For example consider the DDEs

$$(O1) \begin{cases} y'(t) = y^3(t) + y^2(t-1), & t > 0 \\ y(t) = 1, & t \in [-1, 0] \end{cases}$$

$$(O2) \begin{cases} y'(t) = y^2(t) - y^2(t-1), & t > 0 \\ y(t) = 1, & t \in [-1, 0] \end{cases}$$

Applying our result, the problem $(O1)$ blows-up in finite time, while we can see directly that the problem $(O2)$ admits the constant $y \equiv 1$ as a solution on $[-1, \infty[$.

However the case where f is dissipative (good) and g is source is direct.

The following shows that the term g alone could never produce finite time blow-up:

Proposition 4 *Assume that f is a dissipative term, i.e. $xf(x) \leq 0$ for all x. Then for all $\phi \in C$ and all g, the problem (2) admits a global solution.*

Proof. First of all, observe that this condition on f means that for all z_0, z globally exists. Indeed multiplying the equation (1) by z we get $(z^2)' \leq 0$ and so z globally exists.

Now, the idea is that on each interval of length r the term $g(y_r)$ is known: For $t \in [0, r[$ the problem is to show that

$$y'(t) = f(y(t)) + g(\phi(t-r))$$

with the initial data $y(0) = \phi(0)$ exists on $[0, r]$.

Assume that $T_{max} < +\infty$. Then multiplying equation (2) by $y(t)$ we get

$$y'(t)y(t) = y(t)f(y(t)) + y(t)g(y(t-r)) \leq y(t)g(y(t-r)).$$

Set $M := \max\{|g(y(s))|;\ s \in [-r, T_{max} - r]\}$ then for all $t \in [0, T_{max}[$ we have

$$y'(t)y(t) \leq My(t)$$

so

$$\left(y^2(t)\right)' \leq 2M\sqrt{1 + y^2(t)}.$$

By the comparison principle, this implies $0 \leq y^2(t) \leq u(t)$, where $u(t)$ is the solution of the problem $u'(t) = 2M\sqrt{1+u^2}$, $u(0) \geq y(0)$. Since u globally exists we deduce that y^2 and hence y is bounded on $[0, T_{max}[$, which gives a contradiction. □

In the following we give two propositions on the case where f and g are of opposite signs.

Proposition 5 *Assume that $f \geq 0$, $g \leq 0$ and for some z_0, the solution of the problem (1) exists globally. Then for all $\phi \in C$ with $\phi(0) \leq z_0$, problem (2) admits a global solution.*

Proof. By the hypotheses on f and g we have

$$g(y(t-r)) \leq y'(t) \leq f(y(t))$$

on $[0, T_{max}[$. So by the comparison principle we get

$$y(0) + \int_0^t g(y(t-s))\, ds \leq y(t) \leq z(t)$$

where z is the solution of problem (1). Thus if $T_{max} < \infty$, y is bounded which gives a contradiction. □

By the same method we get

Proposition 6 *Assume that $f \leq 0$, $g \geq 0$ and for some z_0, the solution of problem (1) exists globally. Then for all $\phi \in C$, with $\phi(0) \geq z_0$, problem (2) admits a global solution.*

3 Blow-up phenomena for equation (7)

In this section we study the particular equation

$$\begin{cases} y'(t) &= |y(t)|^p - |y(t-r)|^q, \quad t > 0 \\ \\ y(\tau) &= \phi(\tau), \qquad\qquad\qquad \tau \in [-r, 0] \end{cases} \tag{7}$$

with $p, q > 0$. What is the influence of q with respect to p? As we saw, problem ($O2$) has a global solution. Here also the initial data seems to have some influence.

Proposition 7 *Let p, $q > 0$.*

1. *Assume that $p > Max(q, 1)$ and ϕ satisfying $\phi(0) \geq |\phi(t)|^{q/p}$ for all $t \in [-r, 0]$, $\phi(0) \geq 1$, $\phi(0) > |\phi(-r)|^{q/p}$; then the solution of problem (7) blows-up in finite time.*

2. *Assume that $p = q > 1$ and ϕ is a positive C^1 function satisfying $\phi(0) > \phi(-r)$, $\phi(t) \leq \phi(0)$, $\phi'(t) \leq \phi'(0)$ and $\phi'(0) = \phi^p(0) - \phi^p(-r)$. Then the problem (7) blows-up in finite time.*

3. *Assume that $1 < p < q$, and there exists $\varepsilon \in (0, 1)$, $\alpha \in (0, 1]$, and ϕ is such that*

   $$|\phi|^q(t) \leq (1 - \varepsilon)\phi^p(0) \qquad \textit{for all } t \in [-r, (\alpha - 1)r], \tag{8}$$

   and

   $$\phi(0) \geq \left(\frac{\varepsilon}{\alpha(p-1)r}\right)^{\frac{1}{p-1}}. \tag{9}$$

   Then the solution of problem (7) blows-up in finite time.

4. *Assume that $p \leq 1$. Then for all $\phi \in C$, problem (7) admits a global solution.*

Proof. 1. Let T_{max} be the maximal time of existence. By the conditions on ϕ, $y' > 0$. To see this and since $y'(0) > 0$, let $t_1 := \sup\{t \geq 0; y'(s) > 0; \forall s \in [0, t]\}$. If $t_1 < T_{max}$ then $y'(t_1) = 0$. Now if $t_1 - r < 0$ then $0 = y'(t_1)$ and hence $0 = y(t_1) - |\phi(t_1 - r)|^{\frac{q}{p}}$ which is impossible since $y(t_1) > y(0) = \phi(0) \geq |\phi(t_1 - r)|^{\frac{q}{p}}$. If $t_1 - r \geq 0$ and since $y(t) > y(0) = \phi(0) \geq 1$, then $y(t_1) = |y(t_1 - r)|^{\frac{q}{p}} \leq y(t_1 - r)$ which is impossible since y is strictly increasing on $[0, t_1[$. Consequently y is strictly increasing on $[0, T_{max}[$.

Therefore $y^p(t)-|y(t-r)|^q \geq 0$, $y(t) \geq 1$ for all t and hence $y^q(t)-|y(t-r)|^q \geq 0$ also. Writing

$$y'(t) = y^p(t) - y^q(t) + (y^q(t) - |y(t-r)|^q),$$

we get

$$y' \geq y^p - y^q.$$

Therefore, choosing $t_0 \geq 0$ with $y(t_0) > 1$ we have $y(t) \geq z(t)$ for all $t \geq t_0$ where z is the solution of

$$\left\{ \begin{array}{rcl} z'(t) & = & z^p(t) - z^q(t) \\ z(t_0) & = & y(t_0). \end{array} \right.$$

We conclude using the following lemma.

Lemma 3 *For* $z_0 > 1$, $p > \max(q,1)$, *the solution of the problem*

$$\left\{ \begin{array}{rcl} z'(t) & = & |z(t)|^p - |z(t)|^q, \quad t \geq t_0 \\ z(t_0) & = & z_0, \end{array} \right.$$

is positive and blows-up in finite time.

Proof of the lemma. Following the same idea, $z(t)$ is strictly increasing hence positive and $z(t) > 1$ for all $t \geq 0$. Now since the function $g(t) := z^{p-q}(t)$ is increasing and $g(0) > 1$, there exists $\varepsilon > 0$ such that $g(t) \geq \frac{1}{1-\varepsilon}$ and so $z'(t) \geq \varepsilon z^p(t)$.
We now return back to the proof of Proposition 7.
2. $p = q$. First step: y is strictly increasing. Since $y'(0) = \phi^p(0) - \phi^p(-r) > 0$ set $t_1 := \sup\{t \geq 0;\ y'(s) > 0;\ \text{for all } s \in [0, t_1]\}$. If $t_1 < T_{max}$ then $y'(t_1) = 0$. If $t_1 - r < 0$ then $\phi(0) < y(t_1) = |\phi(t_1 - r)|$ which is impossible, and if $t_1 - r \geq 0$ then $y(t_1) = y(t_1 - r)$ which is impossible also since $y'(t_1 - r) > 0$. Therefore $t_1 = T_{max}$ and y is strictly increasing.
Second step: y is convex. By the equality, $y'(t) = y^p(t) - \phi^p(t-r)$, for all $t \in [0, T[$, with $T := \min(r, T_{max})$, we see that y' is continuously differentiable and

$$y''(t) = py^{p-1}(t)y'(t) - p\phi^{p-1}(t-r)\phi'(t-r).$$

Repeating the same analysis, and by the hypothesis on ϕ, $y''(0) > 0$, let then $t_0 := \sup\{t \geq 0;\ y''(s) > 0;\ \text{for all } s \in [0,t]\} \leq T_{max}$. Let's show that $t_0 = T_{max}$: otherwise $y''(t_0) = 0$ and then $y^{p-1}(t_0)y'(t_0) = y^{p-1}(t_0 - r)y'(t_0 - r)$. If $t_0 - r < 0$ then $y^{p-1}(t_0)y'(t_0) = \phi^{p-1}(t_0 - r)\phi'(t_0 - r)$. We have $\phi'(t_0 - r) \leq \phi'(0) \leq y'(t_0)$ but by hypothesis $\phi(t_0 - r) \leq \phi(0) < y(t_0)$ which gives a contradiction. If $t_0 - r \geq 0$ then $y^{p-1}(t_0)y'(t_0) = y^{p-1}(t_0 - r)y'(t_0 - r)$.

Since y is convex on $[0, t_0]$, y' is increasing then $y'(t_0 - r) \leq y'(t_0)$ and we have $y(t_0 - r) < y(t_0)$ which gives a contradiction. Therefore $t_0 = T_{max}$ and y is convex on the whole interval $[0, T_{max}[$.
This implies that the function $t \longmapsto y^p(t) - y_r^p(t)$ is increasing and so for all $t \geq 0$ we have

$$y^p(t) - y_r^p(t) \geq y^p(0) - y_r^p(0) = \phi^p(0) - \phi^p(-r).$$

Choosing $\varepsilon > 0$ so that $(1-\varepsilon)\phi^p(0) - \phi^p(-r) \geq 0$ then for all t we have $(1-\varepsilon)y^p(t) - y_r^p(t) \geq 0$ and then

$$y'(t) = \varepsilon y^p(t) + [(1-\varepsilon)y^p(t) - y_r^p(t)] \geq \varepsilon y^p(t).$$

We conclude using comparison principle.
3. Suppose that the maximal time of existence, T_{max}, is infinite. Since $|\phi|^q(-r) \leq (1-\varepsilon)\phi^p(0) < \phi^p(0)$, $y'(0) > 0$, set then $T := \sup\{0 \leq T \leq +\infty;\ y'(s) > 0;\ \forall s \in [0, t)\}$. Suppose that $T \leq \alpha r$ then $y'(T) = 0$, i.e. $y^p(0) < |y|^p(T) = |\phi|^q(T - r)$ which contradicts the hypothesis.
Now, since for all $0 \leq t \leq \alpha r$, $y'(t) \geq 0$, we have $(1-\varepsilon)y^p(t) > (1-\varepsilon)\phi^p(0) \geq |\phi|^q(t-r)$, hence $y'(t) = y^p(t) - |\phi|^q(t-r) \geq \varepsilon y^p(t)$. But by the hypothesis (9) on $\phi(0)$, the right hand side of this last inequality blows-up at finite time less than αr.
4. This is a direct consequence of Proposition 5. $\square$

It is easy to verify that if ϕ is convex that it verifies the hypotheses of point 2 of the last proposition:

Corollary 1 *Assume that $p = q > 1$ and ϕ is a convex positive C^1 function satisfying $\phi(0) > \phi(-r)$ and $\phi'(0) = \phi^p(0) - \phi^p(-r)$. Then the solution of problem (7) blows-up in finite time.*

Remarks.

1. The case $1 < p < q$: What are necessary conditions on the initial data ϕ so that the solution exists globally? One can expect periodic solutions.

2. The condition (8) replaces the convexity condition.

3. The condition (9) shows the dependence of the blow-up phenomena on the parameters: As $\alpha \to 0$ or $p \to 1$, then $\phi(0)$ must tends to infinity to obtain finite time blow-up.

4. Roughly speaking, point 3 of the last proposition says that if $\phi(0)$ is larger than ϕ on a neighborhood of $-r$, then we have blow-up.

4 Blow-up phenomena for equations (3) and (4)

Concerning the equation (3) we give:

Proposition 8 *Assume that*

(a) f and g are source terms, i.e. $xf(x) > 0$ and $xg(x) > 0$ for all $x \neq 0$.

(b) For some $z_0 > 0$ the problem (1) blows-up in finite time $T^* < \infty$.

Then for all $\phi \geq 0$ such that $\phi(0) > 0$, $\phi(-r) > 0$, $\phi(0) \geq z_0$, and zeros of ϕ are isolated, the solution of problem (3) blows-up in a finite time T.

Proof. First note that by the same technique $y'(t) \geq 0$ for all t. Indeed since $y'(0) = f(\phi(0))g(\phi(-r)) > 0$, set then $t_1 := \sup\{t \geq 0;\ y'(s) \geq 0 \text{ for all } s \in [0,t]\}$ and let's show that $t_1 = T_{max}$. For this suppose that $t_1 < T_{max}$. Since y is increasing and $y(0) = \phi(0) > 0$ then $y(t) > 0$ for all $t \leq t_1$ and so if $t_1 - r \geq 0$ we get $0 = y'(t_1) = f(y(t_1))g(y(t_1 - r)) > 0$. Thus $t_1 - r < 0$. But in this case $0 = y'(t_1) = f(y(t_1))g(\phi(t_1 - r))$ hence $\phi(t_1 - r) = 0$. Since $t_1 - r$ is an isolated zero of ϕ, we have $\phi(t_1 - r + s) > 0$ for all $s \in]0, \varepsilon[$ for some $\varepsilon > 0$. This implies that $y'(t_1 - r + s) = f(y(t_1 + s))g(\phi(t_1 + s - r)) > 0$, which gives a contradiction.

Now we distinguish two cases:
First case: $\min\{\phi(t);\ t \in [-r, 0]\} > 0$. Setting $m := \min\{g \circ \phi(t - r);\ t \in [0, r]\} > 0$, then since y is increasing we have $y'(t) \geq mf(y(t))$ for all t and so $y(t) \geq z(t)$ where z is the solution of the problem $z'(t) = mf(z(t))$ with $z(0) = z_0 > 0$. This implies that y blows-up in finite time.
Second case: $\min\{\phi(t);\ t \in [-r, 0]\} = 0$. Assume that $T_{max} > r$. Since $y' \geq 0$ and $y(0) = \phi(0) > 0$ then $\min\{y(t - r);\ t \in [r, \min(2r, T_{max})[\} > 0$ and then $m := \min\{g \circ y(t - r);\ t \in [r, \min(2r, T_{max})[\} > 0$. This implies that $y'(t) \geq mf(y(t))$. Thus $y(t) \geq u(t)$, where u is the solution of the problem $u'(t) = mf(u(t))$ with $u(r) = y(r)$. By uniqueness we have $y(t) \geq z(t - r) \geq 0$ since $u(r) = y(r) \geq z_0$. Therefore y blows-up in finite time. □

The condition "ϕ of constant sign", can be omitted, if we assume that g is positive.

Concerning the equation (4) we consider the following Cauchy problem

$$\begin{cases} u' = h(u, \alpha) \\ u(0) = u_0 \end{cases} \tag{10}$$

where the function $(x, \alpha) \longmapsto h(x, \alpha)$ is continuous.

We give the following result

Proposition 9 *Assume that there exists α_0 such that*

(a) For all $x \geq 0$, the function $\alpha \in [\alpha_0, +\infty[\longmapsto h(x, \alpha)$ is non decreasing, the function $x \longmapsto h(x, \alpha)$ is locally Lipschitz and $xh(x, \alpha_0) > 0$ for all $x \neq 0$.

(b) For some $u_0 > 0$ the problem (10) with $\alpha = \alpha_0$ admits finite time blow-up solution.

Then for all $\phi \geq 0$, with $\phi(0) \geq u_0$, $\min \phi \geq \alpha_0$, the problem (4) admits a finite time blow-up solution.

Proof. By the same analysis $y'(t) \geq 0$ for all t, then for all t we have $y(t-r) \geq m_0 := \min \phi$. Now since $\alpha \longmapsto h(x, \alpha)$ is non decreasing we have $y'(t) \geq h(y(t), m_0)$, hence $y(t) \geq z(t)$ where z is the solution of the problem $z'(t) = h(z(t), m_0)$, and $z(0) = u_0$. Since $z'(t) \geq h(z(t), \alpha_0)$, then $z(t) \geq u(t)$, the solution of (10) with $\alpha = \alpha_0$. Since $x \longmapsto h(x, \alpha_0)$ is locally Lipschitz, u blows-up in finite time, and $u_0 > 0$, we get $u(t) > 0$ for all t. Therefore y blows-up in finite time. $\square$

Here also, we can omit the hypothesis "ϕ of constant sign", if we assume that h is positive.

5 Examples

In this section we give additional examples showing the different behavior of the solutions depending on the nature of the functions.

1. As an application of Proposition 1, the solution of the problem

$$\begin{cases} y'(t) = y^3(t) + y^2(t-r)\tanh y(t-r), & t > 0 \\ y(t) = \phi(t) > 0, & t \in [-r, 0] \end{cases}$$

blows-up in finite time.

2. By Proposition 3, the solution of the problem

$$\begin{cases} y'(t) = e^{y(t)} + \sin^2 y(t-r), & t > 0 \\ y(t) = \phi(t), & t \in [-r, 0] \end{cases}$$

where ϕ is arbitrary continuous, blows-up in finite time.

3. By proposition 4, for all $\phi \in C$, the solution of the problem

$$\begin{cases} y'(t) = -y^3(t)\sin\frac{|y(t)|}{1+|y(t)|} + \cos(y(t-r)), & t > 0 \\ y(t) = \phi(t), & t \in [-r, 0] \end{cases}$$

has a global solution.

4. By proposition 9 for all $\phi \in C$, $\phi \geq 0$ ($\alpha_0 = 1$), the solution of the problem

$$\begin{cases} y'(t) = y(t)e^{y(t)y(t-r)}, & t > 0 \\ y(t) = \phi(t), & t \in [-r, 0] \end{cases}$$

blows-up in finite time.

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